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Confounding dynamics ☆

Giacomo Rondina [a,*], Todd B. Walker [b]

[a] *University of California – San Diego, United States of America*
[b] *Indiana University, United States of America*

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Abstract

In the context of a dynamic model with incomplete information, we isolate a novel mechanism of shock propagation. We term the mechanism *confounding dynamics* because it arises from agents' optimal signal extraction efforts on variables whose dynamics—as opposed to super-imposed noise—prevents full revelation of information. Employing methods in the space of analytic functions, we are able to obtain analytical characterizations of the equilibria that generalize the celebrated Hansen-Sargent optimal prediction formula. Our main theorem establishes conditions under which confounding dynamics emerge in equilibrium in general settings. We apply our results to a canonical one-sector real business cycle model with dispersed information. In that setting, confounding dynamics is shown to amplify the propagation of a productivity shock, producing hump-shaped impulse response functions.

JEL classification: E32; D84; C62

Keywords: Dispersed information; Confounding dynamics; Rational expectations

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* Corresponding author.
E-mail addresses: grondina@ucsd.edu (G. Rondina), walkertb@indiana.edu (T.B. Walker).

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1. Introduction

Modeling and seeking to understand economic fluctuations is one of the cornerstones of modern economics. The role of incomplete information in this endeavor was acknowledged very early on by Pigou (1929) and Keynes (1936). Their ideas were first formalized in a rational expectations setting by Lucas (1972, 1975), King (1982) and Townsend (1983b). The underlying theme that ties these papers together is that unresolved uncertainty—in and of itself—can be a source of fluctuation in the economy. This idea has seen a resurgence. Dynamic models with dispersed information are becoming increasingly prominent in several literatures such as asset pricing, optimal policy communication, international finance, and business cycles.[1] Our paper contributes to this literature by introducing a novel mechanism of shock propagation, which we call confounding dynamics, and does so in a manner that permits tractability.

Confounding dynamics arise from optimal prediction (i.e. rational expectations) in which past realizations of economic shocks prevent full revelation of information today, even when an arbitrarily large amount of data is available. Ensuring confounding dynamics emerge in equilibrium amounts to deriving non-invertibility restrictions on the equilibrium system of equations. If this system is non-invertible in current and past observations, agents will never fully unravel the contemporaneous economic shock. Our primary example of Section 5, which is based on the real business cycle model of Lucas (1975), shows that non-invertibility of the exogenous process is not a necessary condition for confounding dynamics. The model's cross-equation restrictions endogenously generate non-invertible representations, even when the exogenous process is always invertible. Confounding dynamics can also persist when the number of observables is equal to the number of shocks and therefore, our approach does not rely on the need to overrun the agent's information set with exogenous noise.

We articulate the idea of confounding dynamics in three steps. First, Section 2 derives an optimal prediction formula under confounding dynamics that extends the celebrated Hansen-Sargent formula, and makes an explicit connection to these dynamics. Subsequently, we demonstrate that this behavior carries over to a generic rational expectations model with dispersed information. Our main theorem contains two equations—one that characterizes the dynamic properties of the equilibrium when confounding dynamics are present and one that derives restrictions that guarantee confounding dynamics are preserved in equilibrium. Finally, we provide economic intuition by introducing confounding dynamics into a standard Real Business Cycle model. This application showcases the central insight coming from our main theorem and the defining property of confounding dynamics. The insight is that permitting information to arise endogenously in the context of a model opens the door to an equilibrium that is usually overlooked when information is exogenously provided to agents. Our analytical representation allows us to carefully show how confounding dynamics interacts with crucial parameters of the model. For example, as the elasticity of substitution increases, endogenous variables become more informative and it is more difficult to maintain confounding dynamics in equilibrium. The defining property of confounding dynamics is an impulse response function that is amplified and more persistent relative to the full information equilibrium. There are two possible shapes of an impulse response to a fundamental

[1] The literature is too voluminous to cite every worthy paper. Recent examples include: Woodford (2003), Pearlman and Sargent (2005), Allen et al. (2006), Bacchetta and van Wincoop (2006), Hellwig (2006), Adam (2007), Gregoir and Weill (2007), Angeletos and Pavan (2007), Kasa et al. (2014), Lorenzoni (2009), Rondina (2009), Angeletos and La'O (2009), Angeletos and La'O (2013), Hellwig and Venkateswaran (2009), Graham and Wright (2010), Nimark (2010), Hassan and Mertens (2011), Benhabib et al. (2015), Huo and Takayama (2016) and Angeletos and Lian (2016).

shock under confounding dynamics: [i.] fluctuations around the full information counterpart that display the "waves of optimism and pessimism" of Pigou (1929); and [ii.] an amplified impulse response function that is hump-shaped. We discuss both scenarios in the context of exogenous signal extraction in Section 2. Section 5 focuses on the latter type of impulse response and argues that confounding dynamics—without additional frictions—can provide the internal propagation necessary to match important moments of the data along the lines discussed in Cogley and Nason (1995).

We solve and analyze the rational expectations equilibrium in the space of analytic functions. This approach has several advantages vis-a-vis standard time-domain methods. For example, as emphasized in Townsend (1983a), equilibria are sought in generic functional spaces spanned by linear combinations of shocks, which allows one to avoid explicitly modeling higher-order belief dynamics. Moreover, the matrix Ricatti equation typical of Kalman filtering is replaced by a more transparent spectral factorization problem. This allows us to solve and analyze the equilibrium in closed form. We are not the first to advocate such an approach. Others, such as Futia (1981), Townsend (1983a), Taub (1989), Kasa (2000), Walker (2007), Rondina (2009), Bernhardt et al. (2010), Kasa et al. (2014), and Huo and Takayama (2016) have used similar techniques to solve dynamic rational expectation models with incomplete information. We contribute to this literature by deriving analytical representations (e.g., generalized Hansen-Sargent formulas) and by providing a systematic treatment of equilibrium conditions in models with *dispersed* information that display confounding dynamics. Futia (1981) and Townsend (1983a) were the first to advocate for the use of analytic functions to solve dynamic rational expectations models with heterogeneous information. Many of the mathematical antecedents of this paper can be found there and in Whiteman (1983). Taub (1989) demonstrates how the algebra associated with dynamic signal extraction (i.e., spectral factorization) is simplified through the analytic function approach. We take advantage of these formulas to completely characterize existence and uniqueness of equilibria in dispersed informational setups. Bernhardt et al. (2010) and Kasa et al. (2014) do not examine models with dispersed information, but show how these methods can be used to help resolve asset pricing anomalies.

2. Prediction with confounding dynamics

To study our primary mechanism, we present a simple version of the prediction problem that operates at the heart of the rational expectations equilibria with confounding dynamics. For the reader unfamiliar with frequency domain methods we provide a primer in Appendix C.

Consider the univariate process specified as

$$s_t = -\lambda\varepsilon_t + \varepsilon_{t-1} = (L-\lambda)\varepsilon_t, \tag{1}$$

where ε_t is a mean-zero, normally distributed variable with variable σ_ε^2. Suppose that the prediction problem is to compute the mean-squared error minimizing prediction for ε_t given that s^t is observed. To fix ideas and foreshadow results, imagine that ε_t is the time-t unobserved innovation in aggregate productivity in the economy, while s_t is the observed market rental rate of physical capital. The prediction problem asks for an estimate of the current productivity innovation using the history of the market rental rate.

To solve the problem, we need to consider two possible cases. If $|\lambda| \geq 1$, the process is deemed fundamental for ε_t using the terminology of Rozanov (1967), which means that the stochastic process (1) is invertible in current and past observables; therefore there exists a linear combination of current and past s_t's that allows the exact recovery of ε_t. Defining the lag operator

$Lx_t = x_{t-1}$, one can easily verify that with $|\lambda| \geq 1$, $L - \lambda$ is an invertible operator, and the optimal prediction corresponds to

$$\mathcal{P}(\varepsilon_t|s^t) = \frac{s_t}{L-\lambda} = -\tfrac{1}{\lambda}\left(s_t + \lambda^{-1}s_{t-1} + \lambda^{-2}s_{t-2} + \lambda^{-3}s_{t-3} + ...\right) = \varepsilon_t, \tag{2}$$

which verifies that the history of s^t contains all the information needed to perfectly know ε_t.

Consider now the case of $|\lambda| < 1$. Clearly, the prediction formula (2) is no longer well defined as the coefficients diverge. In this simple environment, Rozanov (1967) shows that the appropriate factorization requires flipping the root λ outside of the unit circle through the use of a Blaschke factor, which we denote as $\mathcal{B}(L) = (1-\lambda L)/(L-\lambda)$.[2] Applying the Blaschke factor results in the optimal prediction,

$$\mathcal{P}(\varepsilon_t|s^t) = -\frac{\lambda}{1-\lambda L}s_t = -\lambda\left(s_t + \lambda s_{t-1} + \lambda^2 s_{t-2} + \lambda^3 s_{t-3} + ...\right) = -\lambda\left(\frac{L-\lambda}{1-\lambda L}\right)\varepsilon_t. \tag{3}$$

Note that the mean squared forecast error of $\left(1-\lambda^2\right)\sigma_\varepsilon^2 > 0$, demonstrating that as $|\lambda|$ approaches one from below there is exact recovery of ε_t.

When the process is non-invertible, (3) shows that the history of current and past s_t's reveals a particular linear combination of ε_t's. Expanding this last term yields

$$\mathcal{P}(\varepsilon_t|s^t) = \underbrace{\lambda^2\varepsilon_t}_{\text{information}} \quad - \quad \underbrace{(1-\lambda^2)[\lambda\varepsilon_{t-1} + \lambda^2\varepsilon_{t-2} + \lambda^3\varepsilon_{t-3} + \cdots]}_{\text{noise from confounding dynamics}}. \tag{4}$$

information + noise from confounding dynamics

Thus, the noise resulting from confounding dynamics takes an unusual form as it consists of a linear combination of past realizations of ε_t. Expression (4) suggests that the process (1) is informationally equivalent to a noisy signal about ε_t, where the noise is the linear combination of past shocks (in the bracketed term), and the signal-to-noise ratio is measured by λ^2. A λ closer to zero results in less information and more noise but, at the same time, it also makes past shocks less persistent. As $\lambda \to 0$, there is no information in s_t about ε_t and the optimal prediction is 0, the unconditional average. As long as $|\lambda| \in (-1, 1)$, the value of ε_t will *never* be learned and in this sense, the *history* of the fundamental shock acts as a noise shock but (as shown below) has non-standard properties. This is the defining property of confounding dynamics. The shocks are perfectly correlated and no super-imposed noise process is necessary to keep full revelation of information from occurring. An infinite history of past shocks is not sufficient because the dynamic history of the shock confounds agents into making forecast errors that would be persistent under the standard full-information rational expectations case.

[2] Specifically, the Blaschke factor flips the zero from inside the unit circle to outside the unit circle via the transformation

$$(L-\lambda)\left(\frac{1-\lambda L}{L-\lambda}\right)\left(\frac{L-\lambda}{1-\lambda L}\right)\varepsilon_t$$

Note that $\mathcal{B}(L)\mathcal{B}(L)^{-1} = 1$, and therefore, the Blaschke factor does not alter the covariance generating function of the time series.

2.1. Economic interpretation

We now provide some economic intuition as it relates to our signal extraction problem, noting that additional intuition is found in Section 5, where we embed this learning mechanism in a real business cycle model.

Comparing representation (1), which we repeat here for convenience, $s_t = (L - \lambda)\varepsilon_t$, to the fundamental representation used to form the optimal prediction (2), $s_t = (1 - \lambda L)\tilde{\varepsilon}_t$ where $\tilde{\varepsilon}_t = \mathcal{B}(L)^{-1}\varepsilon_t$, we see that information is discounted differently. Under full information (assuming agents observe the underlying shocks directly), last period's shock would be discounted more heavily relative to the contemporaneous shock, recall $|\lambda| < 1$. This discounting is exactly reversed when agents have confounding dynamics (assuming agents only observe s^t) with the contemporaneous shock receiving the more significant discount. Therefore, innovations entering the agents' information sets will be discounted differently from the full-information case when confounding dynamics is operational. The extent of the difference in discounting is dictated entirely by the parameter λ: as λ approaches zero (one), the difference will be large (small).

An alternative interpretation comes from noting that confounding dynamics nests the sticky information setup of Mankiw and Reis (2002). When $\lambda = 0$, innovations are observed by agents with a one-period lag, in accordance with sticky information. One might argue that this assumption is too strong in that agents may not ignore *all* information with a one-period lag. Our representation allows for a more continuous interpretation. As $|\lambda|$ approaches one from below starting from zero, agents become more informed. For $|\lambda| \geq 1$, all information is revealed. In principle, one could estimate this parameter using standard methods in a DSGE model. The estimate of λ would then determine the optimal amount of "stickiness" as dictated by data. Several papers argue that sticky information is a natural setup because it can reconcile macro price rigidity with micro price flexibility [Klenow and Willis (2007)] and survey expectations of inflation [Coibion and Gorodnichenko (2012)]. Our approach suggests there is even more flexibility along this dimension.

Finally, we note that the econometrics literature has seen a renewed interest in identification of vector auto-regressions (VAR) in the presence of non-invertibilities [see, Canova and Sahneh (2017)]. One argument in favor of confounding dynamics is that if econometricians using sophisticated techniques have trouble cleanly identifying shocks to the macroeconomy, agents will most likely suffer from similar identification problems, implying non-invertibilities are more likely than not. In this instance, theory can help with measurement because we, as modelers, can cleanly identify ε_t from $\tilde{\varepsilon}_t$, and can then ask how the economy responds to the structural innovation, ε_t, when agents have incomplete information.

2.2. Connection to standard signal extraction

To make the connection to the standard signal extraction problem more explicit, suppose that agents observe an infinite history of the signal

$$x_t = \varepsilon_t + \eta_t, \tag{5}$$

where $\eta_t \overset{iid}{\sim} N\left(0, \sigma_\eta^2\right)$. The optimal prediction is well known and given by $\mathcal{P}(\varepsilon_t | x^t) = \tau x_t$, where τ is the relative weight given to the signal, $\tau = \sigma_\varepsilon^2 / (\sigma_\varepsilon^2 + \sigma_\eta^2)$. It can be shown[3] that

[3] See Online Appendix B.2 for a proof.

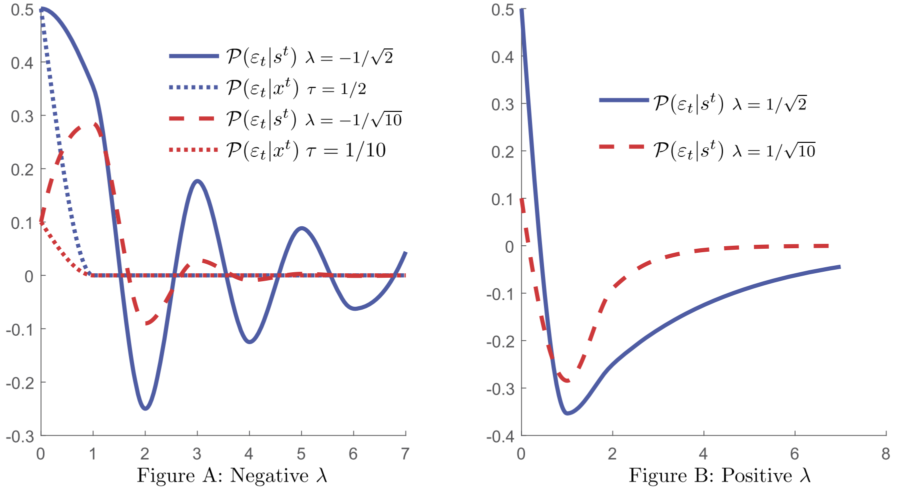


Fig. 1. Panel A: Impulse Responses of x_t and s_t to a one unit change in ε_t for signal-to-noise ratios of $\tau = 1/2$, $\lambda = -1/\sqrt{2}$ (dotted, solid blue) and $\tau = 1/10$, $\lambda = -1/\sqrt{10}$ (dotted, dashed). Panel B: Impulse response of x_t for $\lambda = 1/\sqrt{2}$ (solid) and $\lambda = 1/\sqrt{10}$ (dashed). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

the information content of (1) with $|\lambda| < 1$ is equivalent to (5), where equivalence is defined as equality of variance of the forecast error conditioned on the infinite history of the observed signal, i.e.

$$\mathbb{E}\left[\left(\varepsilon_t - \mathcal{P}\left(\varepsilon_t|s^t\right)\right)^2\right] = \mathbb{E}\left[\left(\varepsilon_t - \mathcal{P}\left(\varepsilon_t|x^t\right)\right)^2\right],$$

when

$$\lambda^2 = \tau. \tag{6}$$

Notice that when the signal-to-noise ratio increases (decreases), this corresponds to a higher (lower) absolute value of λ. In the limit, as $\sigma_\eta^2 \to 0$, then $\lambda^2 \to 1$, which ensures exact recovery of the state in both cases.

While the informational content can be made identical, the dynamics of the two signal extraction problems are very different. To visualize this, we report the impulse response function for the prediction equations that contain confounding dynamics (4) and for the standard signal extraction problem (5) to a one time, one unit increase in ε_t in Fig. 1. We do this for both a low and high value of λ^2 (resp. τ).[4]

Figure A reports a negative value for the non-invertible root λ. Here the impulse response to (4) under-predicts the actual innovation on impact (which is one), with a smaller value of λ under-predicting more significantly. This is due to the first term on the RHS of (4). The same is true for the standard signal extraction formulation (dashed lines). Agents weigh the initial innovation by the signal-to-noise ratio $\tau < 1$ and therefore under-predict on impact. This is where the similarities end. With confounding dynamics, periods two through six show waves of over- and under-prediction relative to the actual realization and relative to the standard signal extraction problem. As discussed above, the current and past innovations will persistently affect the

[4] For aesthetic reasons, the impulse responses are slightly smoothed at turning points.

prediction function several periods beyond impact. This defining characteristic of confounding dynamics leads to the waves of over- and under-reaction. This is in contrast to the full information case and standard signal-extraction case where the impulse response is zero after impact. As already pointed out, the smaller the λ, the larger the noise term in (4), but the less persistent the over- and under-prediction. Thus optimal signal extraction with confounding dynamics generates fluctuations where the full-information and exogenously imposed noise counterparts generate none. Figure B shows that the under- and over-reaction is not the only form of the impulse response under confounding dynamics. A positive value for λ generates an (inverse) hump-shaped impulse response.[5] Again, this can be seen from (4): the under-reaction on impact is the same independent of sign due to the λ^2 term; a positive value for λ implies that the elements of the noise term of (4) all enter with the same sign, causing the impulse to return gradually from below. The larger the value of λ, the more the impulse overshoots. Therefore in either case, confounding dynamics adds persistence to the impulse where traditional signal extraction would not.

3. Model, information, and equilibrium

We now model confounding dynamics in a generic rational expectations formulation that permits many interpretations (e.g., monetary model, asset pricing model, etc.). We do this via dispersed information, which introduces well-known difficulties. We lay out a solution strategy and compare that strategy to alternative methodologies.

3.1. Model

We consider models that are populated by a continuum of agents indexed by $i \in [0, 1]$. Let $\mu(i)$ be the density of agent i characterized by the information set at time t, denoted by Ω_{it}. We are interested in the class of models in which the individual optimal choice can be represented by the dynamic expectational difference equation,

$$\phi \mathbb{E}\left[X_{it+1} \middle| \Omega_{it}\right] = \psi(L) X_{it}, \tag{7}$$

where

$$X_{it} \equiv \begin{pmatrix} x_{it} & y_t & \theta_{it} \end{pmatrix}^\top \tag{8}$$

Here $\phi \equiv [\phi_x \; \phi_y \; \phi_\theta]$, is a vector of coefficients, and $\psi(L) \equiv [\psi_x(L) \; \psi_y(L) \; \psi_\theta(L)]$, is a vector of square-summable lag polynomials in non-negative powers of L. x_{it} is the choice variable under the control of the individual agent i; y_t is an endogenous aggregate variable that agents take as given, and θ_{it} is an exogenous stochastic process specified as the sum of an aggregate component θ_t and an i.i.d. individual component v_{it}. Formally

$$\theta_{it} = \theta_t + v_{it}, \quad \text{where} \quad \theta_t = A(L)\varepsilon_t, \tag{9}$$

with $v_{it} \sim \mathcal{N}(0, \sigma_v)$, $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon)$, and $A(L)$ is a square-summable polynomial in non-negative powers of L. Our main theorem will deliver the restrictions on parameters needed to ensure the

[5] In a different setting, Acharya et al. (2017) show that the combination of sentiment shocks and non-invertibilities can generate hump-shaped impulse response functions as well.

equilibrium system of equations is non-invertible in current and past observations; i.e., that confounding dynamics obtains in equilibrium. To close the model we need to specify a relationship between the distribution of x_{it} across agents, and the aggregate y_t. We thus posit that

$$\gamma(L) \int_0^1 X_{it}\mu(i)di = 0, \tag{10}$$

where $\gamma(L) \equiv [\gamma_x(L)\ \gamma_y(L)\ \gamma_\theta(L)]$, is a vector of square-summable finite-degree lag polynomials in non-negative powers of L, and we assume $\gamma_x(L) \neq 0$.[6] As we proceed with the analysis it will be useful to think of equation (7) as representing a demand (or supply) schedule for agent i, and (10) as the relevant market clearing condition. However, the specific form depends on the particular application at hand.

The expectational difference equation (7) is a dispersed information version of the system originally considered by Blanchard and Kahn (1980), and subsequently studied by Uhlig (1999), Klein (2000) and Sims (2002), among others. Dispersed information implies that individual expectations are heterogeneous, which implies that the aggregation in (10) will result in taking an average of expectations. In particular, model (7)-(10) can accommodate both average expectations of aggregate variables and average expectations of individual variables.

3.2. *Information*

In our dispersed information setup, we assume that the information set Ω_{it} of an arbitrary agent i at time t consists of the smallest closed subspace generated by the history of the random variable $\theta_i^t \equiv \{\theta_{it}, \theta_{it-1}, ...\}$, and the history of the aggregate variable $y^t = \{y_t, y_{t-1}, ...\}$. Specifically, $\Omega_t^i = \theta_i^t \vee y^t$, where the operator $\vee$ denotes the span (i.e., the smallest closed subspace which contains the subspaces) generated by the sequences θ_i^t and y^t. This notation simply suggests that expectations will be taken optimally; i.e., they will be consistent with the prediction formulas discussed in Section 2. In a multivariate moving-average setting, the invertible representation achieved via canonical factorization is the smallest closed subspace containing the observables, θ_i^t and y^t (see Hoffman (1962)).

Given (7), x_{it} will be a function of the history of idiosyncratic innovations, v_{it}, and the aggregate innovations, ε_t, namely

$$x_{it} = X(L)\varepsilon_t + V(L)v_{it}. \tag{11}$$

In addition, aggregation implies that y_t is only a function of aggregate innovations, so that

$$y_t = Y(L)\varepsilon_t. \tag{12}$$

The signal structure can be thus represented as

$$\begin{pmatrix} \theta_{it} \\ y_t \end{pmatrix} = \Gamma(L) \begin{pmatrix} \sigma_\varepsilon^{-1}\varepsilon_t \\ \sigma_v^{-1}v_{it} \end{pmatrix}, \quad \Gamma(L) = \begin{bmatrix} A(L)\sigma_\varepsilon & \sigma_v \\ Y(L)\sigma_\varepsilon & 0 \end{bmatrix}. \tag{13}$$

We point out that our information set is in line with the typical information set assumed in the dispersed information rational expectations literature: we provide agents with both an *exogenous*

[6] We make this assumption in order to keep the connection between (7) and (10) non-trivial. Allowing for $\gamma_x(L) = 0$, would imply that y_t is directly determined by the process θ_t, and, as a consequence, it would enter (7) as an exogenous variable, essentially duplicating the role of θ_t in that equation.

signal about the aggregate unobserved state (θ_{it}), and an *endogenous* signal that is determined in equilibrium (y_t). The analytical convenience of the signal structure (13), for our purposes, is that the invertibility of the matrix $\Gamma(L)$ hinges only upon the zeros of $Y(L)$. At the same time, the structure imposes analytical discipline that is uncommon in the literature: the endogenous signal y_t can reveal perfectly the underlying state, under the appropriate parametrization of model (7)-(10). Thus, we aim at establishing both the degree to which information remains incomplete in equilibrium, along with the more standard existence and uniqueness conditions.

3.3. Examples

We pause briefly here to note that our general setup can handle a wide variety of models. Appendix B.7 carefully walks readers through four such examples: an RBC model, the asset pricing model of Singleton (1987), a model with Calvo pricing and a New Keynesian Phillips Curve, and the classical monetary models of inflation of Cagan (1956). Of course, this list is not exhaustive but there are two common characteristics in all of the examples: [i.] shocks are Gaussian and [ii.] the model can be written in a linear form. As with nearly all papers in this literature, our analysis relies on linear projections being consistent with optimal conditional expectations, which necessitates [i] and [ii].

3.4. Equilibrium definition

Uncertainty is assumed to be driven by Gaussian innovations, which, together with linearity, implies that conditional expectations are computed as optimal linear projections. We thus have

$$\mathbb{E}\left(X_{it+1}\middle|\Omega_{it}\right)=\mathcal{P}\left[X_{it+1}\middle|\Omega_{it}\right], \tag{14}$$

and can apply the Wiener-Kolmogorov prediction formula (see Appendix C) to compute conditional expectations. We are now ready to define a Rational Expectations Equilibrium for model (7)-(10).

Definition REE. *A Rational Expectations Equilibrium (REE) is a stochastic process for* $\{X_{it}, i \in [0,1]\}$ *and a stochastic process for the information sets* $\{\Omega_{it}, i \in [0,1]\}$ *such that: (i) each agent* i*, given her information set, forms expectations according to* (14)*; (ii)* $\{X_{it}, i \in [0,1]\}$ *satisfies conditions* (7)-(10).

The REE can be summarized by two statements: (a) given a distribution of information sets, there exists a market clearing distribution $\{X_{it}, i \in [0,1]\}$ determined by each agent i's optimal prediction conditional on the information sets; (b) given a distribution $\{X_{it}, i \in [0,1]\}$, there exists a distribution of information sets that provides the basis for optimal prediction. Both statements (a) and (b) must be satisfied by the same distribution $\{X_{it}, i \in [0,1]\}$ and the same distribution of information sets *simultaneously* in order to satisfy the requirements of a REE. This dual fixed point condition is standard in rational expectations with potentially heterogeneously informed agents and when endogenous variables convey information [see, Radner (1979) as an early example].

3.5. Weighted sum of expectations

Before discussing our solution methodology, we give a brief overview of the typical approach to solve model (7)-(10), which consists of two steps. The first step is to iteratively substitute the endogenous variables x_{it+j} and y_{t+j} forward by leading (7) j periods forward and aggregating over agents. The end result is expressions for x_{it} and y_t, that are a function of expectations of the exogenous variable θ_t at all future horizons. The second step is then to compute those expectations, which is non-trivial due to the fact that the law of iterated expectations may not be operational. Most of the work that uses this approach rely on numerics to calculate these expectations.[7]

Consider the expression for $\phi_x \mathbb{E}_{it}(x_{it+1})$. Through forward substitution, this expression contains the term $\phi_x \mathbb{E}_{it+1}(x_{it+2})$, which in turn contains θ_{t+2}. It follows that the law of iterated expectations (LIE) applies in this context so that $\phi_x^2 \mathbb{E}_{it}\mathbb{E}_{it+1}(\theta_{t+2}) = \phi_x^2 \mathbb{E}_{it}(\theta_{t+2})$, and aggregation implies $\phi_x^2 \bar{\mathbb{E}}_t(\theta_{t+2})$ for $j = 2$. Intuitively, in each round of the iterative substitutions there are terms where agent i is taking expectations of both her own future expectations and of future average expectations. The law of iterated expectations applies to the former, so that the order of expectations is reduced, but not to the latter.[8] It should be evident at this point that the second step required by the canonical approach—computing closed form solutions for the expectations of arbitrary order—is a daunting task under dispersed information (for more details on this, see Appendix B.5). As already remarked and discussed thoroughly in the next section, we approach the solution from a different angle.

3.6. Solution methodology

Our aim is to characterize a REE equilibrium for model (7)-(10) with confounding dynamics. The critical requirement for confounding dynamics to emerge is that the information matrix $\Gamma(L)$, (13), must be non-invertible at a $\lambda \in (-1, 1)$. However, there is no guarantee that this condition will hold. Consistent with the intuition of Townsend (1983a), our approach is to formulate a guess for the endogenous variables that follows a generic polynomial in the underlying shocks, and then derive conditions on the *exogenous* parameters that yield non-invertibility in equilibrium.

Our main theorem (Theorem 1) and corollary in Section 5 restricts attention to functional forms with exactly one λ inside the unit circle. The solution procedure described below is consistent with this restriction. However, equilibrium conjectures of functional forms with multiple λ's inside the unit circle can be entertained within the procedure described below with appropriate modifications. Appendix B.4 shows how to solve the exogenous signal extraction problem with multiple roots inside the unit circle, which provides a road map for how to modify Steps 1-4 below to solve for the rational expectations equilibrium in that case.

The following steps describe our procedure when looking for an equilibrium with confounding dynamics.

1. Specify the guesses for x_{it} and y_t as generic polynomials of underlying shocks

$$x_{it} = X(L)\varepsilon_t + V(L)v_{it}, \quad \text{and} \quad y_t = Y(L)\varepsilon_t, \tag{15}$$

where y_t has confounding dynamics, so that

[7] Nimark (2010) and Melosi (2017) are recent examples of sophisticated numerical methods to characterize equilibria with dispersed information.

[8] Mechanically, whether LIE applies or not at each iteration depends on the position of ϕ_x in the coefficients of the polynomial $(\phi_x + \phi_y)^j$, i.e. on the set of permutations of size j of ϕ_y and ϕ_x with repetition. For instance, for the case of $j = 2$, the set of terms that multiply ψ_θ are $(\phi_y^2 + \phi_y\phi_x)\bar{\mathbb{E}}_t\bar{\mathbb{E}}_{t+1}(\theta_{t+2}) + (\phi_x\phi_y + \phi_x^2)\bar{\mathbb{E}}_t(\theta_{t+2})$.

$$Y(\lambda) = 0, \quad \text{for} \quad \lambda \in (-1, 1). \tag{16}$$

2. Given the signal matrix $\Gamma(L)$, obtain the so-called canonical factorization $\Gamma^*(L)$ under (16) (see Appendix C for a discussion of the canonical factorization).
3. Use $\Gamma^*(L)$ together with the guesses in (15) to obtain the conditional expectations in (7) via the Wiener-Kolmogorov prediction formula.
4. Aggregate over agents according to (10) and use the relationship between $X(L)$ and $Y(L)$ to substitute $X(L)$ with $Y(L)$ in (7). Both the right hand side and the left hand side will now be lag polynomial operators in ε_t and v_{it}, and will thus provide the fixed point conditions for $Y(L)$ and $V(L)$.
5. Derive conditions on exogenous parameters so as to ensure that a determinate stationary solutions exists, and that there exists a $|\lambda| < 1$, verifying (16). Once $Y(L)$ is solved for, use (10) to recover $X(L)$.

Note that at no point in the solution procedure one needs to worry about higher-order expectations. The so-called "higher-order thinking" that complicates the iterative approach outlined in Section 3.5 is implicit in how the guess (15) combines with the information matrix $\Gamma(L)$ to provide a closed form for the first order expectations in (7). As recognized by Townsend (1983a), by guessing a generic lag polynomial, the higher-order beliefs are built into the guess and we do not have to track these terms explicitly, although higher-order beliefs can be backed out of the solution in closed form. The same solution procedure is followed when we solve for an equilibrium with full information, with the only difference that condition (16) is not imposed, and thus does not have to be verified, and the signal matrix $\Gamma(L)$ corresponds to full information.

4. Equilibrium with confounding dynamics

This section establishes the main result of the paper: the existence of a rational expectations equilibrium with confounding dynamics in a dispersed information environment.

4.1. Equilibrium with confounding dynamics: main theorem

In this section we state our main Theorem, which provides conditions under which a REE with Confounding Dynamics exists. As stated in Section 3.2, we specify the information set as

$$\Omega_{it} = \theta_i^t \vee y^t \tag{17}$$

Agents thus observe the entire history of the exogenous process θ_{it} up to time t, together with the history of the aggregate variable y_t. In addition, the model equations (7)-(10) are both common knowledge across agents.

An important building block in the statement of our main theorem is the full information benchmark solution which we denote by, $x_{it} = \mathcal{X}(L)\varepsilon_t + \mathcal{V}(L)v_{it}$, and $y_t = \mathcal{Y}(L)\varepsilon_t$, where $\mathcal{X}(L)$, $\mathcal{V}(L)$ and $\mathcal{Y}(L)$ are square-summable lag polynomial in non-negative powers of L. In the full information solution, each agent i is provided with the entire history of shocks, ε_t and v_{it}, up to time t. The derivation of the full information solution is reported in Appendix A.1. Here we point out that to ensure uniqueness (determinacy) of a full information solution, the characteristic polynomials of the expectational difference equations for $\mathcal{V}(L)$ and $\mathcal{Y}(L)$, which are defined respectively by $\phi_x(L) \equiv \phi_x - \psi_x(L)$, and $\Phi(L) \equiv \phi_x(L) + \phi_y - \psi_y(L)L$, must satisfy a standard regularity condition, which corresponds to our Assumption 1.

Assumption 1. *The polynomials $\phi_x(L)$ and $\Phi(L)$ each have exactly one root inside the unit circle.*

It is important to note that this assumption does not correspond to a special case, nor is overly restrictive. It amounts to restricting equilibria to stationary processes both within the cross-section and time series dimensions of the model. Requiring $\Phi(L)$ to have one root inside the unit circle is the standard assumption necessary to yield a unique rational expectations equilibrium (e.g., Sims (2002)) and it immediately implies that $\Phi(L)$ can be factorized as

$$\Phi(L) = (\zeta - L)\tilde{\Phi}(L), \tag{18}$$

where $|\zeta| < 1$, and $\tilde{\Phi}(L)$ has no roots inside the unit circle. If the polynomial had no such roots inside the unit circle, the RE equilibrium would not be unique; and if the polynomial had multiple roots inside the unit circle, no stationary equilibrium would exist. Similarly, requiring $\phi_x(L)$ to have one root inside the unit circle ensures that the cross-sectional distribution is well defined at any point in time. In the equilibrium with confounding dynamics, the expectational difference equations for $V(L)$ and $Y(L)$ contain the same characteristic polynomials $\phi_x(L)$ and $\Phi(L)$ of the full information benchmark, and we thus also impose the regularity conditions of Assumption 1 in Theorem 1.

Recall that the key requirement in solving for an equilibrium with confounding dynamics is that there exists a $\lambda \in (-1, 1)$ such that $Y(\lambda) = 0$. We are interested in finding restrictions on exogenous parameters so that a λ that satisfies such requirement exists. Theorem 1 states our main result.

Theorem 1. *Consider model (7)-(10) with Assumption 1. Let the information sets be specified as in $\Omega_{it} = \theta_i^t \vee y^t$. There exists a Rational Expectations Equilibrium with Confounding Dynamics of the form, $y_t = Y(L)\varepsilon_t$, with*

$$Y(L) = \mathcal{Y}(L) - \big(1 - \tau(\lambda)\big)\big(1 - \lambda^2\big)\frac{\mathcal{A}(\lambda)}{(1 - \lambda L)\tilde{\Phi}(L)}, \tag{19}$$

if there exists a $\lambda \in (-1, 1)$ that solves

$$\mathcal{Y}(\lambda)\tilde{\Phi}(\lambda) = \big(1 - \tau(\lambda)\big)\mathcal{A}(\lambda), \tag{20}$$

where $\mathcal{Y}(L)$ is the full information solution, $\tau(\lambda) \equiv \frac{A(\lambda)^2\sigma_\varepsilon^2}{A(\lambda)^2\sigma_\varepsilon^2 + \sigma_v^2}$, $\mathcal{A}(\lambda)$ is a function of λ that depends only on exogenous parameters, and $Y(L)$ in (19) has a zero inside the unit circle equal to λ.

Proof. See Appendix A.2. □

Theorem 1 provides sufficient conditions for the existence of an equilibrium that belongs to a class in which $Y(L)$ takes a functional form with *exactly one zero* inside the unit circle, that is $Y(L) = (L - \lambda)G(L)$, where $G(L)$ is a stationary lag polynomial with no zeros inside the unit circle. Within the "exactly one zero" class, condition (20) might be satisfied by more than one numerical value for λ. Each value corresponds to a legitimate equilibrium within the class once substituted into (19) because the fixed-point conditions would be satisfied. These equilibria are indexed by information, since each distinct numerical value of λ reflects how much information is revealed in equilibrium. The notion of "multiplicity" in this scenario is not related to the well-known indeterminacy criteria in rational expectations models, where a continuum of equilibria exists. In fact, Assumption 1 rules out that type of multiplicity here. Theorem 1 does allow for more than one rational expectations equilibrium in the "exactly one zero" class, and such equilibria are "locally unique" in the sense that small perturbations of the information sets will not lead to an alternative λ-value and therefore will not diverge to an alternative rational expectations equilibrium.

4.2. *Outline of proof*

The proof consists of four steps and can be found in its entirety in Appendix A.2. We briefly discuss each step, relegating tedious algebra to the appendix.

STEP 1: FACTORIZATION We operationalize the key requirement that $Y(\lambda) = 0$ for $\lambda \in (-1, 1)$ by specifying a guess of the form $Y(L) = (L - \lambda)G(L)$, where $G(L)$ has no zeros inside the unit circle. The first step in the proof is to then use the equilibrium guess to derive the canonical factorization for the information set, which can be written as

$$\begin{pmatrix} \theta_{it} \\ y_t \end{pmatrix} = \begin{bmatrix} A(L)\sigma_\varepsilon & \sigma_v \\ (L-\lambda)G(L)\sigma_\varepsilon & 0 \end{bmatrix} \begin{pmatrix} \tilde{\varepsilon}_t \\ \tilde{v}_{it} \end{pmatrix}, \tag{21}$$

where $\varepsilon_t = \sigma_\varepsilon \tilde{\varepsilon_t}$, $v_{it} = \sigma_v \tilde{v_{it}}$, is a convenient normalization so that the variance-covariance matrix of the innovations vector is the identity matrix. The following lemma gives the canonical factorization for $\Gamma(L)$.

Lemma 1. *The canonical factorization* $\Gamma^*(z)\Gamma^*(z^{-1})^T$ *of the variance-covariance matrix* $\Gamma(z)\Gamma(z^{-1})^T$, *is given by*

$$\Gamma^*(z) = \frac{1}{\sqrt{A(\lambda)^2\sigma_\varepsilon^2 + \sigma_v^2}} \begin{bmatrix} A(z)A(\lambda)\sigma_\varepsilon^2 + \sigma_v^2 & \sigma_\varepsilon\sigma_v \frac{1-\lambda z}{z-\lambda}\big(A(z) - A(\lambda)\big) \\ A(\lambda)\sigma_\varepsilon^2(z-\lambda)G(z) & \sigma_\varepsilon\sigma_v G(z)(1-\lambda z) \end{bmatrix}. \tag{22}$$

Proof. See Appendix A.2. □

STEP 2: EXPECTATIONS Equipped with the canonical factorization (22), we next derive the three expectational terms: $\mathbb{E}_{it}(x_{it+1})$, $\mathbb{E}_{it}(y_{t+1})$, and $\mathbb{E}_{it}(\theta_{it+1})$ from direct application of the Wiener-Kolmogorov prediction formula. The last two follow directly,

$$\mathbb{E}_{it}\begin{pmatrix} \theta_{it+1} \\ y_{t+1} \end{pmatrix} = \big[L^{-1}\Gamma^*(L)\big]_+ \Gamma^*(L)^{-1} \begin{pmatrix} \theta_{it} \\ y_t \end{pmatrix}.$$

However, the term $\mathbb{E}_{it}(x_{it+1})$, is substantially more involved to derive, due to the fact that the correlation between x_{it+1} and θ_{it} exists not only because they both depend on ε_t, but they also both depend on v_{it}. Formally, the application of the Wiener-Kolmogorov formula leads to

$$\mathbb{E}_{it}(x_{it+1}) = \Big[L^{-1}g_{x_i,(\theta_i,y)}(L)\big(\Gamma^*(L^{-1})^T\big)^{-1}\Big]_+ \Gamma^*(L)^{-1} \begin{pmatrix} \theta_{it} \\ y_t \end{pmatrix},$$

where $g_{x_i,(\theta_i,y)}(L)$ is the variance-covariance generating function between x_i and the information set. Given the equilibrium guess, such a function takes the form

$$g_{x_i,(\theta_i,y)}(L) = \big[\, X(L)A(L^{-1})\sigma_\varepsilon^2 + V(L)\sigma_v^2 \quad X(L)(L^{-1}-\lambda)G(L^{-1})\sigma_\varepsilon^2 \,\big].$$

A bit of algebra gives

$$L^{-1}g_{x_i,(\theta_i,y)}(L)\big(\Gamma^*(L^{-1})^T\big)^{-1}$$
$$=\Big[L^{-1}\big(V(L)\sigma_v^2+X(L)\sigma_\varepsilon^2 A(\lambda)\big)\quad \sigma_\varepsilon\sigma_v L^{-1}\tfrac{1-\lambda L}{L-\lambda}\Big(X(L)-V(L)A(\lambda)\Big)\Big].$$

Acknowledging that the terms have the usual principal part around $L=0$ *and* around $L=\lambda$, it follows that

$$\begin{aligned}\mathbb{E}_{it}(x_{it+1}) =&L^{-1}\Big[X(L)-X(0)\Big]\varepsilon_t-\big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[X(\lambda)-X(0)\\&-\big(V(\lambda)-V(0)\big)A(\lambda)\Big]\varepsilon_t\\&+L^{-1}\Big[V(L)-V(0)\Big]v_{it}+\tfrac{\tau(\lambda)}{A(\lambda)}\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[X(\lambda)-X(0)\\&-\big(V(\lambda)-V(0)\big)A(\lambda)\Big]v_{it}.\end{aligned}\tag{23}$$

Step 3: Fixed Point. Next, we need to derive and check the fixed-point conditions. This amounts to algebraic manipulations that serve to get the model in the form such that existence and uniqueness criteria can be invoked, as well as the condition $Y(\lambda)=0$. Here we report the part of the proof that focuses on making sure that the fixed point in information is established. The proof consists in checking that when the equilibrium coefficients are evaluated using a λ that solves (20), there are no other points at which $Y(L)$ vanishes inside the unit circle. More precisely, it has to be that there is no $\xi\neq\lambda$ that solves

$$\mathcal{Y}(\xi)\tilde{\Phi}(\xi)=(1-\tau(\lambda))(1-\lambda^2)\frac{\mathcal{A}(\lambda)}{1-\lambda\xi},\tag{24}$$

such that $|\xi|\in(-1,1)$. If such a ξ existed, the information conveyed by y_t in equilibrium would be inconsistent with the information used to compute the expectations that are part of the fixed point. To see this, suppose that for a given λ that solves (20), a $|\xi|<1$ that satisfies (24) exists. The $Y(L)$ solution computed using that λ would have, by construction, another zero at ξ. If we denote that solution by $\tilde{Y}(L)=\tilde{G}(L)(L-\lambda)(L-\xi)$, the factorization (22) would only remove the zero associated with λ so that Step 1 above would give

$$\Gamma^*(L)=\frac{1}{\sqrt{A(\lambda)^2\sigma_\varepsilon^2+\sigma_v^2}}\begin{bmatrix}A(L)A(\lambda)\sigma_\varepsilon^2+\sigma_v^2 & \sigma_\varepsilon\sigma_v\frac{1-\lambda L}{L-\lambda}\big(A(L)-A(\lambda)\big)\\ A(\lambda)\sigma_\varepsilon^2(L-\lambda)(L-\xi)\tilde{G}(L) & \sigma_\varepsilon\sigma_v\tilde{G}(L)(L-\xi)(1-\lambda L)\end{bmatrix}.\tag{25}$$

Note that the determinant of $\Gamma^*(L)$ in (25) vanishes at $L=|\xi|<1$, so the factorization will result in expectations that are conditioned on an information set that is inconsistent with the information revealed in equilibrium. In other words, for the specific λ under consideration, the fixed point in information at the heart of Theorem 1 would fail to be verified.

If no equilibria with confounding dynamics with exactly one zero can be found, one can modify the initial guess and consider $N>1$ roots inside the unit circle, looking then for a condition analogue to (20) to deliver exactly N solutions. We restrict our attention to $N=1$ for simplicity and because the full description of the space of REE with confounding dynamics is beyond the scope of this paper, but we hope it is clear that our methods extend to the more general case.

Step 4: No Information from the Model The last thing to check to complete the proof is to ensure that there is no information that is transmitted by a clever manipulation of the model conditions – which are part of the information set of the agents – combined with the knowledge of the history of θ_{it} and y_t. For instance, suppose that the market clearing condition (10) is specified so that $\int_0^1 x_{it}\mu(i)di = y_t$, which means that y_t is the aggregate of x_{it}, then this would imply $X(L) = Y(L)$, which would result in $x_{it} - y_t = V(L)v_{it}$. Because rational agents know this, they know that the difference $x_{it} - y_t$ is just a linear combination of the individual innovations v_{it}. It follows that they could, in principle, back out the realizations of v_{it}'s by inverting $V(L)$. More generally, the link between $X(L)$ and $Y(L)$ due to (10) can be used by rational agents to obtain additional information on the underlying innovations. For this not to happen, if one augments the information set of the agents by $x_{it} - y_t$, the information matrix must still be non-invertible at λ. The following Lemma shows that this is indeed the case for the equilibrium of Theorem 1.

Lemma 2. *In the equilibrium with confounding dynamics of Theorem 1, consider the augmented information matrix $\tilde{\Gamma}(L)$, where*

$$\begin{pmatrix} \theta_{it} \\ y_t \\ x_{it} - y_t \end{pmatrix} = \tilde{\Gamma}(L) \begin{pmatrix} \varepsilon_t \\ v_{it} \end{pmatrix} = \begin{bmatrix} A(L) & 1 \\ Y(L) & 0 \\ X(L) - Y(L) & V(L) \end{bmatrix} \begin{pmatrix} \varepsilon_t \\ v_{it} \end{pmatrix}. \tag{26}$$

The 2-by-2 minors of $\tilde{\Gamma}(L)$ all vanish at λ.

Proof. See Appendix A.2. □

The form of (19) is intuitive when contrasted with the full information counterpart. The standard Hansen-Sargent formula subtracts off the particular linear combination of *future* values of ε_t that minimize the agent's forecast error. As described in Section 2, confounding dynamics implies that a particular linear combination of *past* values of ε_t are never revealed to the agent. In order to make a direct comparison to the full-information case transparent, set $\gamma_y(L) = \gamma_x(L)$, $\psi_x(L) = 1$, $\psi_y(L) = 0$, $\phi_\theta = 0$ and $\psi_\theta(L) = -1$. According to Theorem 1, the solution under confounding dynamics can be written as

$$y_t = \sum_{j=0}^{\infty} \zeta^j \theta_{t+j} - A(\zeta) \sum_{j=1}^{\infty} \zeta^j \varepsilon_{t+j} - \big(1 - \tau(\lambda)\big)(1 - \lambda^2)\mathcal{A}(\lambda) \sum_{j=0}^{\infty} \lambda^j \varepsilon_{t-j}. \tag{27}$$

The first two components on the right-hand side of (27) give the standard (full-information) Hansen-Sargent formula. The third component—represented by the weighted sum $\sum_{j=0}^{\infty} \lambda^j \varepsilon_{t-j}$—arises due to confounding dynamics and is similar to the prediction formula of Section 2. Agents do not observe the linear combination of shocks weighted by λ. Conditioning down implies that this linear combination will (optimally) be subtracted from the Hansen-Sargent full-information equilibrium. The relevance of the unknown past depends on the imprecision of the private signal θ_{it}, measured by $1 - \tau(\lambda)$; the imprecision stemming from confounding dynamics, measured by $1 - \lambda^2$; and the fixed point constant $\mathcal{A}(\lambda)$.

Equation (20) provides the condition for the existence of equilibrium (19). It is obtained by evaluating the right-hand side of (19) at λ and setting it equals to zero. By doing so, (20) is ensuring that once the conditioning down due to confounding dynamics is taken into account, the λ responsible for such conditioning down must indeed be a point in which the equilibrium function is non-invertible. Condition (20) takes an intuitive form from an informational point of view. Note first that the LHS, $\mathcal{Y}(\lambda)\tilde{\Phi}(\lambda)$, corresponds to the moving average part of the full information solution evaluated at λ (a complete derivation of the full-information counterpart is presented in the Appendix A.1). Suppose for a moment that the RHS of (20) is set to zero. If a $|\lambda| \in (0, 1)$ satisfying the condition existed, it would mean that the equilibrium with confounding dynamics would take the same form as the full information equilibrium $\mathcal{Y}(L)$. However, equation (27) shows that in presence of confounding dynamics the unknown past must be subtracted from the full information equilibrium, which would make the full information solution $\mathcal{Y}(L)$ inconsistent with confounding dynamics. The implication of this observation is that whenever the RHS of (20) is made small enough, an equilibrium with confounding dynamics may fail to exist. In particular, as the noise-to-signal ratio in private information $\sigma_v/\sigma_\varepsilon$ declines, the signal-to-noise ratio, $\tau(\lambda)$, gets closer to one, and eventually leads to non-existence of an equilibrium with confounding dynamics.

We finally note that the autoregressive factor in (19), $1/(1-\lambda L)$, injects into the equilibrium dynamics of y_t the waves of over- and under-reaction or the hump-shaped imupulse depicted in Fig. 1, which are the hallmark of signal extraction under confounding dynamics. In Section 5, in the context of a real business cycle model, we provide a description of how economic incentives can combine with the signal extraction under non-invertibility to deliver the fixed-point condition (20), and a hump-shaped response to shocks exclusively due to confounding dynamics.

5. Application: business cycle with confounding dynamics

In this section we apply our results to a model of business cycle fluctuations driven by productivity shocks. The purpose of this section is to analytically demonstrate the confounding dynamics mechanism within a well established framework. To achieve this goal, we work within a linearized model reminiscent of the islands model of Lucas (1975). We motivate this section with two observations: First, note that this application allows us to demonstrate that the sufficient conditions for confounding dynamics are non-empty. Second, a common criticism of many models that follow the Lucas tradition is that agents cannot see economy-wide prices: if they could, then they could infer fundamentals perfectly and there would not be any confusion in equilibrium. Our setup does not suffer from this criticism.[9]

The economy consists of a continuum of islands indexed by $i \in [0, 1]$. Each island is inhabited by an infinitely-lived representative household, and by a representative firm, also indexed by i. Household i supplies labor services exclusively to firm i in a decentralized competitive labor market or, equivalently, workers cannot move across islands. Households supply labor inelastically to firms, and the labor supply is normalized to 1. Households own capital in the economy, which is rented out to firms in a centralized spot market. Firms use capital and labor to produce output, also supplied in a centralized competitive spot market. Households derive utility from consuming the output good. Output is produced by firm i according to a Cobb-Douglas technology with capital and labor inputs – with income shares α, and $1-\alpha$ respectively, and total factor of productivity that is firm-specific and denoted by $e^{a_{it}}$, where

$$a_{it} = a_t + v_{it}.$$

The term a_t is common across all the islands, while v_{it} is a productivity component that is specific to island i. In what follows, we consider a log-linearized version of the model with full capital

[9] We are not unique in this respect, see for instance Amador and Weill (2010).

depreciation and constant elasticity of intertemporal substitution, denoted by $\eta > 0$.[10] Household i sets consumption intertemporally according to the Euler equation

$$\mathbb{E}_{it}\big(c_{it} - c_{it+1} + \eta r_{t+1}\big) = 0. \tag{28}$$

The intertemporal budget constraint is

$$(1 - \beta\alpha)c_{it} + \alpha\beta k_{it+1} = (1 - \alpha)w_{it} + \alpha r_t - \alpha k_{it}, \tag{29}$$

where k_{it+1} is the capital stock that household i is carrying into period $t + 1$, w_{it} is the wage rate, r_t is the rental rate of capital, and $\beta \in (0, 1)$ is the subjective discount factor. The island-specific wage rate is given by, $w_{it} = \frac{1}{1-\alpha}(a_{it} - \alpha r_t)$. Aggregate capital is defined as $k_{t+1} \equiv \int_0^1 k_{it+1}\mu(i)di$, and market clearing implies an interest rate

$$r_t = a_t - (1 - \alpha)k_t. \tag{30}$$

Using the household's budget constraint at t and at $t + 1$ to get expressions for c_{it} and c_{it+1}, and leading (30) one period forward, one can substitute (28) into the Euler to obtain a second-order difference equation for capital k_{it+1}

$$\alpha\beta\mathbb{E}_{it}\big(k_{it+2}) + \eta(1 - \alpha\beta)\mathbb{E}_{it}\big(r_{t+1}\big) - \mathbb{E}_{it}\big(a_{it+1}\big) = \alpha(1 + \beta)k_{it+1} - \alpha k_{it} - a_{it}, \tag{31}$$

which completely characterizes the equilibrium. As remarked in Section 3.3, the model maps into our general setting by specifying $x_{it} = k_{it+1}$, $y_t = r_t$, and $\theta_{it} = a_{it}$.

Finally, we assume that total factor productivity that is common across islands follows the $AR(1)$ process

$$a_t = \rho a_{t-1} + \varepsilon_t, \tag{32}$$

so that $A(L) = \frac{1}{1-\rho L}$, and with $\rho \in [0, 1]$. Note that there are *no* moving average components in this process, and therefore it is always invertible. It cannot be the source of confounding dynamics. They must emerge naturally from interactions within the model.

Full Information We first derive the full information ($\Omega_{it} = v_i^t \vee \varepsilon^t$) solution for aggregate capital and the interest rate. The full-information guess for island-specific capital is given by $k_{it+1} = \mathcal{K}(L)\varepsilon_t + \mathcal{V}(L)v_{it}$. From (30), the interest rate is immediately determined by $r_t = \mathcal{R}(L)\varepsilon_t$ and where

$$\mathcal{R}(L) = A(L) - (1 - \alpha)\mathcal{K}(L)L. \tag{33}$$

The characteristic polynomial associated with equation (31) can be determined as

$$\Phi(L) = \alpha\beta - \big(\eta(1 - \alpha\beta)(1 - \alpha) + (1 + \beta)\alpha\big)L + \alpha L^2 = \alpha(\zeta - L)(\beta/\zeta - L). \tag{34}$$

Given that α (capital's share of production) and β (subjective discount factor) are both less than one, (34) contains one root inside the unit circle (ζ) and one outside (β/ζ), and their product is always equal to β. Following the steps outlined in Section A.1, the full information equilibrium for capital can be derived as the $AR(2)$ process

[10] The fully specified model and the derivation of the log-linearization are reported in the Online Appendix B.6.

$$\mathcal{K}(L) = \frac{\frac{\zeta}{\alpha\beta}(1+\kappa)}{(1-\rho L)(1-\frac{\zeta}{\beta}L)}, \tag{35}$$

and the interest rate takes an $ARMA(2,1)$ form

$$\mathcal{R}(L) = \frac{1 - \frac{\zeta}{\alpha\beta}(1+(1-\alpha)\kappa)L}{(1-\rho L)(1-\frac{\zeta}{\beta}L)}, \tag{36}$$

where $\kappa \equiv \frac{\rho(1-\zeta)(\alpha\beta/\zeta-1)}{(1-\rho\zeta)(1-\alpha)}$.

The standard assumptions in the RBC model imply that productivity affects the interest rate contemporaneously, while investment in capital affects the interest rate with a one period lag. The consequence of this timing assumption is that r_t features a moving average component which we know, from our analysis, can have important informational consequences. Suppose that the moving average in r_t was non-invertible. If agents were asked to extract the history of ε_t based solely on data from r_t, they would face the signal extraction problem described in Section 2. In particular, they would not be able to recover the exact history of ε_t. In the full information equilibrium reported above, agents are assumed to directly observe ε_t in every period, and so the equilibrium dynamics are consistent with the information used to compute expectations even if r_t itself is non-invertible. However, what if we modify the information available to the agents by removing the direct observation of the shocks? How would the equilibrium change? Theorem 1 can be readily applied to address these questions, to which we now turn.

Confounding Dynamics The first step in applying Theorem 1 is to specify the agents' information set. Because households participate in two competitive markets every period – the labor market and the rental market for capital – they observe the island-specific wage rate w_{it}, and the rental rate r_t. The observation of w_{it} and r_t implies that household i can always back out a_{it} at time t through the expression for w_{it} reported above. As a consequence, observing the prices of labor and capital is equivalent to the information set

$$\Omega_{it} = a_i^t \vee r^t. \tag{37}$$

We also assume that households cannot observe the aggregate capital k_t, so to avoid the full revelation of a_t, and thus v_{it}, which would be implied by (30).[11] Following Theorem 1, existence of confounding dynamics requires that the process for $r_t = R(L)\varepsilon_t$, has the following property,

$$R(\lambda) = 0, \tag{38}$$

for a $\lambda \in (-1,1)$. A direct application of Theorem 1 leads to the following corollary.

Corollary 1. *Consider the Real Business Cycle model* (30)-(32). *Let the information sets be specified as in* (37). *There exists a Rational Expectations Equilibrium with Confounding Dynamics of the form, $k_{t+1} = K(L)\varepsilon_t$, and $r_t = R(L)\varepsilon_t$, with*

$$K(L) = \mathcal{K}(L) - \big(1-\tau(\lambda)\big)\mathcal{C}(\lambda)\frac{(1-\frac{\zeta}{\beta}\lambda)(1-\lambda^2)}{(1-\frac{\zeta}{\beta}L)(1-\lambda L)}, \tag{39}$$

and $R(L) = A(L) - (1-\alpha)K(L)L$, if there exists a $\lambda \in (-1,1)$, that solves

[11] There are many other information structures that would preserve confounding dynamics in this setting and would be consistent with the general specification of Section 3.1.

$$\mathcal{R}(\lambda) = -\big(1-\tau(\lambda)\big)(1-\alpha)\mathcal{C}(\lambda)\lambda, \tag{40}$$

where $\mathcal{C}(\lambda) \equiv \frac{\left(\frac{1-\beta}{1-\rho\lambda}\right)((1-\lambda\beta)\lambda-\tau(\lambda)(1-\lambda^2)\beta)+(1-\tau(\lambda))(1-\lambda^2)\beta((1-\rho\zeta)\kappa/\rho-\zeta)+(1-\lambda\beta)\lambda\kappa/\rho}{\lambda\alpha(\lambda-\beta/\zeta)((1-\lambda\beta)(\lambda-\zeta)-(1-\lambda^2)\tau(\lambda)(\beta-\zeta))}$. $\mathcal{K}(L)$ *and* $\mathcal{R}(L)$ *are as in* (35) *and* (36), $\tau(\lambda) \equiv \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2+(1-\rho\lambda)^2\sigma_v^2}$, *and* $R(L)$ *has a zero inside the unit circle equal to* λ.

While the functional forms of equations (39)–(40) have the same general structure as Theorem 1 (and same interpretation), the context of the application allows us to gain additional insights into the existence and behavior of an equilibrium with confounding dynamics.[12]

Table 1 reports the endogenous values of λ computed solving (40); "none" indicates that there is no $\lambda \in (-1, 1)$ that solves (40). In Panel 1, the elasticity of substitution, η, is held fixed at 1 – corresponding to log utility – and the private signal precision, $\sigma_v/\sigma_\varepsilon$, is changed from very informative (column (a)) to very uninformative (column (c)). An equilibrium with confounding dynamics exists when the private signal is uninformative: column (c) with $\lambda = 0.73$. Intuitively, if the private signal is very informative, agents will rely strongly on their private information in forming their beliefs about aggregate productivity, which, in turn, will make the interest rate more informative. In Panel 2, $\sigma_v/\sigma_\varepsilon$ is held fixed at 2, and the elasticity of substitution is changed from a low level (0.5 in column (a)), to a high level (2 in column (c)). In this case, the equilibrium with confounding dynamics only exists when the elasticity of substitution is sufficiently low: column (a), with $\lambda = 0.44$. From the full information equilibrium (35) we see that a lower elasticity of substitution implies a more sluggish response of capital as agents are less willing to substitute consumption for investment.[13] In the presence of incomplete information, a similar sluggish adjustment prevents capital, and thus the interest rate, to correctly reflect the underlying changes in fundamentals. As the elasticity of substitution is increased, the more reactive response of capital results in the interest rate dynamics fully revealing the fundamentals.

We conclude the analysis by using the case in column (c), in Panel 1 of Table 1 to study the qualitative effects of confounding dynamics on capital and the interest rate. Fig. 2 shows the response of capital, k_{t+1}, and the interest rate, r_t, to a persistent unitary positive shock to aggregate productivity a_t under full information (dashed lines) and confounding dynamics (plain lines). Under full information, capital increases at impact and steadily climbs towards a new persistent level (recall that $\rho \approx 1$ in this example). The interest rate increases at impact because capital is fixed at first while productivity is higher. Subsequently, the interest rate steadily declines because of the increased capital accumulation which reduces the marginal product of capital.

In the equilibrium with confounding dynamics the impulse responses are markedly different. As shown by the solid lines in Fig. 2, the response of capital is amplified for every t and displays a hump-shaped pattern, with the peak reached in period 1 and a persistent slow decline towards the full information long-run level. The interest rate at impact is equal to the full-information

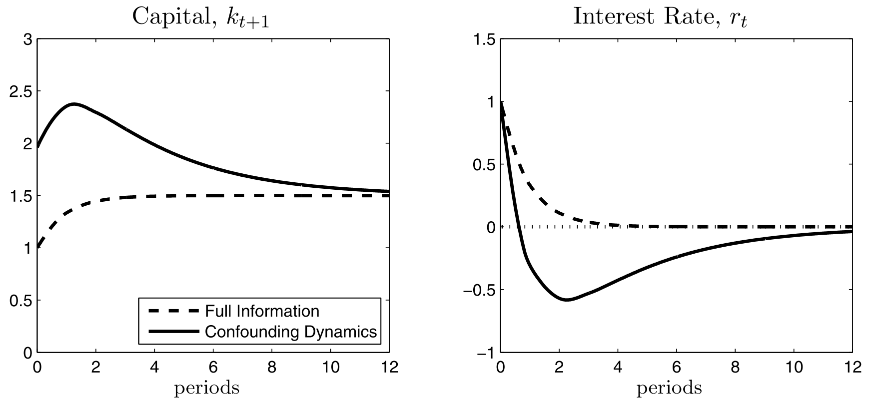
[12] Note that, as it is the case for Theorem 1, the corollary looks for an equilibrium functional form with exactly one root (λ) inside the unit circle. In our numerical analysis, it can happen that condition (40) is satisfied by more than one *numerical value* for λ. For each individual numerical value, we verify that the initial conjectured equilibrium functional form holds, so we confirm that we have identified a rational expectations equilibrium (i.e. we implement the fixed point check described in Step 3 of the sketch of the proof of Theorem 1). We have also verified that the qualitative properties of the equilibrium are the same across numerical values and the overall message of our results do not change.

[13] To see this, note that (34) implies $\frac{\partial\zeta}{\partial\eta} > 0$, which in turn implies $\frac{\partial\mathcal{K}(0)}{\partial\eta} > 0$ for $\rho \approx 1$.

Table 1
Existence of Equilibrium with Confounding Dynamics.

Panel 1: $\eta = 1$	(a)	(b)	(c)
Noise-Signal Ratio, $\sigma_v/\sigma_\varepsilon$	0.1	1	4
Confounding Dynamics, λ	none	none	0.73
Panel 2: $\sigma_v/\sigma_\varepsilon = 2$	(a)	(b)	(c)
Elasticity of Substitution, η	0.5	1	2
Confounding Dynamics, λ	0.44	none	none

Existence of Equilibrium with Confounding Dynamics for numerical values of the noise-to-signal ratio in a_{it}, $\sigma_v/\sigma_\varepsilon$, and the elasticity of intertemporal substitution, η. The rest of the parameters are set at $\beta = 0.985$, $\alpha = .33$, $\rho \approx 1$. The entry "none" indicates that there is no $\lambda \in (-1, 1)$ that solves (40).



Impulse response of Capital, k_{t+1}, and Interest Rate, r_t, under Full Information (dash-line) and Confounding Dynamics (plain-line). The parameter values are $\eta = 1$, $\beta = 0.985$, $\alpha = 0.33$, $\rho \approx 1$, and $\sigma_v/\sigma_e = 4$. For the Confounding Dynamics equilibrium, $\lambda = 0.73$.

Fig. 2. Impulse Response of Capital and Interest Rate.

case because capital is fixed, but it drops in negative territory in the subsequent periods because of the larger response of capital.

The intuition as of why capital displays an amplified response under confounding dynamics can be found in how capital behaves to ensure that the interest rate is non-invertible at λ. When $\eta = 1$, and $\rho \approx 1$, one can show that[14]

$$\mathcal{R}(L) \approx \frac{1}{(1 - \alpha L)}, \tag{41}$$

which means that the full information interest rate decays gradually towards zero after impact as capital gradually climbs towards a very persistent higher level. In the equilibrium with confounding dynamics, a moving average component with root λ appears in the process for the interest rate, which requires the interest rate to overshoot into negative territory after impact, similarly

[14] To see this note that when $\eta = 1$, $\zeta/\beta = \alpha$, and $\kappa = 0$, so the expression immediately follows from (36) when one recognizes that the ratio $(1 - L)(1 - \rho L)$ cancels for $\rho \approx 1$. Also note that this relationship only holds under full information.

to Panel B in Fig. 1. For that to be the case, the interest rate must decline more than the full-information case one period after impact, which, given the timing of the model, can happen only with a higher response of capital at impact. Using (41) together with (40) it is possible to show that the difference between the dynamic response of capital across the two equilibria is

$$K(L) - \mathcal{K}(L) \approx \frac{(1-\lambda^2)}{(1-\alpha)\lambda(1-\alpha L)(1-\lambda L)}. \tag{42}$$

The hump shape with peak at period 1 emerges because, $\alpha + \lambda > 1$, and, $\alpha^2 + \lambda^2 + \alpha\lambda < \alpha + \lambda$, in our numerical example. Intuitively, the persistence in the interest rate dynamics, measured by α, combines with the persistence due to signal extraction from the interest rate, measured by λ, and they initially reinforce each other before eventually declining.

The role of the informativeness of private signal, a_{it}, measured by $\sigma_v/\sigma_\varepsilon$, is also crucial in sustaining the amplified response and thus non-invertibility. Based just on their private signal, optimal signal extraction would instruct agents to be conservative in estimating the innovation to aggregate productivity, which would result in lower investment at the individual agent's level compared to full information and thus aggregate under-reaction of investment. However, the behavior of the interest rate under confounding dynamics changes the average *predicted* innovation in a_t. If agents observe a large drop in the interest rate after impact, their signal extraction effort leads them to rationally infer that the aggregate productivity shock is larger than what their private signal alone would suggest. In this sense, the interest rate dynamics, when used to extract information about the innovation in productivity, acts as a perceived positive *aggregate* innovation in productivity. If the private signal is sufficiently uninformative, the perceived innovation remains consistent with rational expectations, and an equilibrium with confounding dynamics is established.

Our application starkly showcases the central insight coming from Theorem 1: allowing for the endogeneity of signals in a dynamic context opens the door to a set of equilibria that are usually overlooked when information is exogenously provided to the agents. Fig. 2 shows that equilibria with confounding dynamics can display a qualitative behavior of key aggregate variables that is interesting and promising for quantitative applications. The shape and size of the response is determined by the assumption that we look at equilibria with only one non-invertible root λ. However, richer non-invertible conditions – such as ones with multiple roots, conjugate pairs, etc. – would result in richer dynamics that would ensure a better fit of data (we explore a simple prediction example with multiple roots in Appendix B.4). Finally, in order to keep things analytically tractable and transparent, we have assumed away additional sources of frictions, thereby limiting the potential of the model to provide quantitatively significant results. However, we envision a richer environment with several types of frictions, such as financial frictions – which are likely to introduce stronger sensitivity of allocations to the interest rate, or exogenous noisy signals, but where confounding dynamics remain a major determinant of equilibrium behavior.

6. Concluding comments

As we have shown, confounding dynamics injects persistence into impulse response functions. These interesting dynamics are generated from a simple and optimal learning mechanism that can be easily applied to any dynamic setting. Future work will seek to better understand the empirical properties of confounding dynamics by incorporating them into real and nominal business cycle models designed to be taken to data. Theoretical results of Section 5 and preliminary empirical results show much promise. Future work will also seek to show an equivalence between the analytic function approach advocated here and the more familiar time-domain approach. Contrasting these approaches in a side-by-side fashion will help to highlight the benefits of the analytic function approach while demystifying certain aspects of it.

Appendix A. Proofs

A.1. Full information solution

The proof of Theorem 1 makes use of the full information solution of (10)-(7). We report the derivation of the full information solution here for completeness. We define as Full Information the case when every agent is endowed with perfect knowledge of the aggregate and her own idiosyncratic innovations history up to time t. Denoting the full information set by $\tilde{\Omega}_{it}$, the set is formally specified as

$$\tilde{\Omega}_{it} = v_i^t \vee \varepsilon^t. \tag{A.1}$$

Here, and in the following analysis, we assume that agents know that the equilibrium relationship is given by (7)-(10). We begin by guessing that the solution takes the form, $x_{it} = \mathcal{X}(L)\varepsilon_t + \mathcal{V}(L)v_{it}$, and $y_t = \mathcal{Y}(L)\varepsilon_t$, where $\mathcal{X}(L)$, $\mathcal{V}(L)$ and $\mathcal{Y}(L)$ are square-summable lag polynomial in non-negative powers of L. Under full information, direct application of the Wiener-Kolmogorov formula (see Appendix C) provides expressions for the relevant expectational terms,

$$\mathbb{E}_{it}(x_{it+1}) = [\mathcal{X}(L) - \mathcal{X}(0)]L^{-1}\varepsilon_t + [\mathcal{V}(L) - \mathcal{V}(0)]L^{-1}v_{it}, \tag{A.2}$$

$$\mathbb{E}_{it}(y_{t+1}) = [\mathcal{Y}(L) - \mathcal{Y}(0)]L^{-1}\varepsilon_t, \tag{A.3}$$

$$\mathbb{E}_{it}(\theta_{t+1}) = [A(L) - A(0)]L^{-1}\varepsilon_t. \tag{A.4}$$

The fixed point condition under full information can be found by substituting (A.2)-(A.4) into (7), so that

$$\begin{aligned}
&\phi_x\big[\mathcal{X}(L) - \mathcal{X}(0)\big]L^{-1}\varepsilon_t + \phi_x\big[\mathcal{V}(L) - \mathcal{V}(0)\big]L^{-1}v_{it} + \phi_y\big[\mathcal{Y}(L) - \mathcal{Y}(0)\big]L^{-1}\varepsilon_t \\
&\quad + \phi_\theta\big[A(L) - A(0)\big]L^{-1}\varepsilon_t \\
&\quad = \psi_x(L)\mathcal{X}(L)\varepsilon_t + \psi_x(L)\mathcal{V}(L)v_{it} + \psi_y(L)\mathcal{Y}(L)\varepsilon_t + \psi_\theta(L)A(L)\varepsilon_t + \psi_\theta(L)v_{it}.
\end{aligned} \tag{A.5}$$

This equation defines a fixed point condition for $\mathcal{V}(L)$ with all the terms that multiply v_{it}. Collecting terms that multiply v_{it}, multiplying both sides by L and rearranging we get

$$\mathcal{V}(L)\big(\phi_x - \psi_x(L)L\big) = \phi_x\mathcal{V}(0) + \psi_\theta(L)L. \tag{A.6}$$

Note that $\phi_x(L) \equiv \phi_x - \psi_x(L)L$, which, under Assumption 1, has exactly one zero inside the unit circle, which we term ζ_x. We thus pick $\mathcal{V}(0)$ to remove such zero by setting

$$\phi_x\mathcal{V}(0) + \psi_\theta(\zeta_x)\zeta_x = 0. \tag{A.7}$$

Solving for $\mathcal{V}(0)$, substituting back into (A.6) one finally obtains

$$\mathcal{V}(L) = \frac{\psi_\theta(L)L - \psi_\theta(\zeta_x)\zeta_x}{\phi_x(L)}. \tag{A.8}$$

We now focus on the fixed point for $\mathcal{Y}(L)$ and $\mathcal{X}(L)$. As remarked in the text, the fixed point condition does not feature any components of $\mathcal{V}(L)$, so that one does not need to solve for the latter to obtain the former. To proceed with the solution there are two possibilities: solve for $\mathcal{Y}(L)$ and then recover $\mathcal{X}(L)$, or viceversa. In general, both routes are possible, but there are situations in which one direction is substantially easier than the other. This depends on whether $\gamma_x(0) \neq 0$ or $\gamma_y(0) \neq 0$. We report here both cases. We first consider the case that works whenever $\gamma_x(0) \neq 0$. We begin by manipulating condition (10) to get the following relationship between $\mathcal{X}(L)$ and $\mathcal{Y}(L)$,

$$\mathcal{X}(L) = \tilde{\gamma}_y(L)\mathcal{Y}(L) + \tilde{\gamma}_\theta(L)A(L), \tag{A.9}$$

where $\tilde{\gamma}_y(L) = -\frac{\gamma_y(L)}{\gamma_x(L)}$, and $\tilde{\gamma}_\theta(L) = -\frac{\gamma_\theta(L)}{\gamma_x(L)}$. Using (A.9) to substitute for terms featuring $\mathcal{X}(L)$ in (A.5) one obtains

$$\begin{aligned}
&\phi_x\big[\tilde{\gamma}_y(L)\mathcal{Y}(L) - \tilde{\gamma}_y(0)\mathcal{Y}(0)\big]L^{-1}\varepsilon_t + \phi_x\big[\tilde{\gamma}_\theta(L)A(L) - \tilde{\gamma}_\theta(0)A(0)\big]L^{-1}\varepsilon_t \\
&\qquad + \phi_x\big[\mathcal{V}(L) - \mathcal{V}(0)\big]L^{-1}v_{it} + \phi_y\big[\mathcal{Y}(L) - \mathcal{Y}(0)\big]L^{-1}\varepsilon_t + \phi_\theta\big[A(L) - A(0)\big]L^{-1}\varepsilon_t \\
&\quad = \psi_x(L)\tilde{\gamma}_y(L)\mathcal{Y}(L)\varepsilon_t + \psi_x(L)\tilde{\gamma}_\theta(L)A(L)\varepsilon_t \\
&\qquad + \psi_x(L)\mathcal{V}(L)v_{it} + \psi_y(L)\mathcal{Y}(L)\varepsilon_t + \psi_\theta(L)A(L)\varepsilon_t + \psi_\theta(L)v_{it}.
\end{aligned} \tag{A.10}$$

Taking all the terms that multiply ε_t in (A.10), multiplying by L both sides and rearranging, one gets

$$\mathcal{Y}(L)\Phi(L) = \mathcal{Y}(0)\big(\phi_x\tilde{\gamma}_y(0) + \phi_y\big) - \xi_y(L), \tag{A.11}$$

where

$$\xi_y(L) \equiv \big(\phi_x - \psi_x(L)L\big)\tilde{\gamma}_\theta(L)A(L) + \big(\phi_\theta - \psi_\theta(L)L\big)A(L) - \big(\phi_x\tilde{\gamma}_\theta(0) + \phi_\theta\big)A(0). \tag{A.12}$$

Under Assumption 1, $\Phi(L)$ has exactly one zero inside the unit circle, denoted by ζ, which means that we can choose $\mathcal{Y}(0)$ to remove such zero. We thus set

$$\mathcal{Y}(0)\big(\phi_x\tilde{\gamma}_y(0) + \phi_y\big) - \xi_y(\zeta) = 0. \tag{A.13}$$

Solving for $\mathcal{Y}(0)$, substituting into (A.11) and rearranging, one finally gets

$$\mathcal{Y}(L) = \frac{\xi_y(\zeta) - \xi_y(L)}{\Phi(L)}. \tag{A.14}$$

The expression for $\mathcal{X}(L)$ can then be recovered using (A.9). Next we consider the case that works whenever $\gamma_y(0) \neq 0$. We begin by manipulating condition (10) to get the following relationship between $\mathcal{X}(L)$ and $\mathcal{Y}(L)$,

$$\mathcal{Y}(L) = \hat{\gamma}_x(L)\mathcal{X}(L) + \hat{\gamma}_\theta(L)A(L), \tag{A.15}$$

where $\hat{\gamma}_x(L) = -\frac{\gamma_x(L)}{\gamma_y(L)}$, and $\hat{\gamma}_\theta(L) = -\frac{\gamma_\theta(L)}{\gamma_y(L)}$. Using (A.15) to substitute for terms featuring $\mathcal{Y}(L)$ in (A.5) one obtains

$$\begin{aligned}
&\phi_x\big[\mathcal{X}(L) - \mathcal{X}(0)\big]L^{-1}\varepsilon_t + \phi_x\big[\mathcal{V}(L) - \mathcal{V}(0)\big]L^{-1}v_{it} \\
&\qquad + \phi_y\big[\hat{\gamma}_x(L)\mathcal{X}(L) - \hat{\gamma}_x(L)\mathcal{X}(0)\big]L^{-1}\varepsilon_t \\
&\qquad + \phi_y\big[\hat{\gamma}_\theta(L)A(L) - \hat{\gamma}_\theta(L)A(0)\big]L^{-1}\varepsilon_t + \phi_\theta\big[A(L) - A(0)\big]L^{-1}\varepsilon_t \\
&\quad = \psi_x(L)\mathcal{X}(L)\varepsilon_t + \psi_x(L)\mathcal{V}(L)v_{it} + \psi_y(L)\hat{\gamma}_x(L)\mathcal{X}(L)\varepsilon_t \\
&\qquad + \psi_y(L)\hat{\gamma}_\theta(L)A(L)\varepsilon_t + \psi_\theta(L)A(L)\varepsilon_t + \psi_\theta(L)v_{it}.
\end{aligned} \tag{A.16}$$

Taking all the terms that multiply ε_t in (A.16), multiplying by L both sides and rearranging, one gets

$$\mathcal{X}(L)\Phi_x(L) = \mathcal{X}(0)\big(\phi_x + \phi_y\hat{\gamma}_y(0)\big) - \xi_x(L), \tag{A.17}$$

where

$$\Phi_x(L) = \phi_x + \phi_y\hat{\gamma}_x(L) - \psi_x(L)L - \psi_y(L)\hat{\gamma}_x(L)L, \tag{A.18}$$

and

$$\xi_x(L) \equiv \big(\phi_y - \psi_y(L)L\big)\hat{\gamma}_\theta(L)A(L) + \big(\phi_\theta - \psi_\theta(L)L\big)A(L) - \big(\phi_y\hat{\gamma}_\theta(0) + \phi_\theta\big)A(0). \tag{A.19}$$

Analogously to Assumption 1, let us assume that $\Phi_x(L)$ has exactly one zero inside the unit circle, denoted by $\hat{\zeta}$, which means that we can choose $\mathcal{X}(0)$ to remove such zero. We thus set

$$\mathcal{X}(0)\big(\phi_x + \phi_y\hat{\gamma}_y(0)\big) - \xi_x(\hat{\zeta}) = 0. \tag{A.20}$$

Solving for $\mathcal{X}(0)$, substituting into (A.17) and rearranging, one finally gets

$$\mathcal{X}(L) = \frac{\xi_x(\hat{\zeta}) - \xi_x(L)}{\Phi_x(L)}. \tag{A.21}$$

The expression for $\mathcal{Y}(L)$ can then be recovered using (A.15).

A.2. *Proof of Theorem 1*

STEP 1: FACTORIZATION We operationalize the key requirement that $Y(\lambda) = 0$ for $\lambda \in (-1, 1)$ by specifying a guess of the form $Y(L) = (L - \lambda)G(L)$, where $G(L)$ has no zeros inside the unit circle. The first step in the proof is to then use the equilibrium guess to derive the canonical factorization for the information set, so that the Wiener-Kolmogorov formula (see Appendix C) be applied. The information set can be written as

$$\begin{pmatrix} \theta_{it} \\ y_t \end{pmatrix} = \begin{bmatrix} A(L)\sigma_\varepsilon & \sigma_v \\ (L-\lambda)G(L)\sigma_\varepsilon & 0 \end{bmatrix} \begin{pmatrix} \tilde{\varepsilon}_t \\ \tilde{v}_{it} \end{pmatrix}, \tag{A.22}$$

where $\varepsilon_t = \sigma_\varepsilon\tilde{\varepsilon}_t$, $v_{it} = \sigma_v\tilde{v_{it}}$, is a convenient normalization so that the variance-covariance matrix of the innovations vector is the identity matrix. It follows that

$$\Gamma(L) = \begin{bmatrix} A(L)\sigma_\varepsilon & \sigma_v \\ (L-\lambda)G(L)\sigma_\varepsilon & 0 \end{bmatrix}. \tag{A.23}$$

The following Lemma shows the canonical factorization for $\Gamma(L)$.

Lemma A3. *The canonical factorization $\Gamma^*(z)\Gamma^*(z^{-1})^T$ of the variance-covariance matrix $\Gamma(z)\Gamma(z^{-1})^T$, where $\Gamma(z)$ is defined in (A.23), is given by*

$$\Gamma^*(z) = \frac{1}{\sqrt{A(\lambda)^2\sigma_\varepsilon^2 + \sigma_v^2}} \begin{bmatrix} A(z)A(\lambda)\sigma_\varepsilon^2 + \sigma_v^2 & \sigma_\varepsilon\sigma_v\frac{1-\lambda z}{z-\lambda}\big(A(z) - A(\lambda)\big) \\ A(\lambda)\sigma_\varepsilon^2(z-\lambda)G(z) & \sigma_\varepsilon\sigma_v G(z)(1-\lambda z) \end{bmatrix}. \tag{A.24}$$

Proof. Using Rozanov (1967) procedure, $\Gamma^*(z)$ is computed as

$$\Gamma^*(z) = \Gamma(z)W_\lambda B_\lambda(z), \tag{A.25}$$

where

$$W_\lambda = \frac{1}{\sqrt{A(\lambda)^2\sigma_\varepsilon^2+\sigma_v^2}}\begin{bmatrix} A(\lambda)\sigma_\varepsilon & -\sigma_v \\ \sigma_v & A(\lambda)\sigma_\varepsilon \end{bmatrix}, \quad \text{and} \quad B_\lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-\lambda z}{z-\lambda} \end{bmatrix}. \tag{A.26}$$

The form of W_λ is obtained by application of Lemma C1 in Appendix C. Solving out the matrix multiplication after some algebra one obtains (A.24). □

STEP 2: EXPECTATIONS Equipped with the canonical factorization (A.24), we next derive the three expectational terms: $\mathbb{E}_{it}(x_{it+1})$, $\mathbb{E}_{it}(y_{t+1})$, and $\mathbb{E}_{it}(\theta_{it+1})$ (recall that $\mathbb{E}_{it}(\theta_{it+1}) = \mathbb{E}_{it}(\theta_{t+1})$). The second and third in the list are given by

$$\mathbb{E}_{it}\begin{pmatrix} \theta_{it+1} \\ y_{t+1} \end{pmatrix} = \left[L^{-1}\Gamma^*(L)\right]_+ \Gamma^*(L)^{-1}\begin{pmatrix} \theta_{it} \\ y_t \end{pmatrix}. \tag{A.27}$$

Recalling that $\left[L^{-1}\Gamma^*(L)\right]_+ = \left[\Gamma^*(L) - \Gamma^*(0)\right]L^{-1}$, and defining $\tau(\lambda) = \frac{A(\lambda)^2\sigma_\varepsilon^2}{A(\lambda)^2\sigma_\varepsilon^2+\sigma_v^2}$ one gets

$$\begin{aligned} \mathbb{E}_{it}(\theta_{t+1}) &= \left[A(L) - A(0)\right]L^{-1}\varepsilon_t - \left(1-\tau(\lambda)\right)\frac{1-\lambda^2}{\lambda(1-\lambda L)}\left[A(\lambda) - A(0)\right]\varepsilon_t \\ &\quad - \tau(\lambda)\frac{1-\lambda^2}{\lambda(1-\lambda L)}\left[1 - \frac{A(0)}{A(\lambda)}\right]v_{it}, \end{aligned} \tag{A.28}$$

$$\begin{aligned} \mathbb{E}_{it}(y_{t+1}) &= \left[(L-\lambda)G(L) + \lambda G(0)\right]L^{-1}\varepsilon_t - \left(1-\tau(\lambda)\right)\frac{1-\lambda^2}{(1-\lambda L)}G(0)\varepsilon_t \\ &\quad + \tau(\lambda)\frac{1-\lambda^2}{(1-\lambda L)}\frac{G(0)}{A(\lambda)}v_{it}. \end{aligned} \tag{A.29}$$

The term $\mathbb{E}_{it}(x_{it+1})$, is substantially more involved to derive, due to the fact that the correlation between x_{it+1} and θ_{it} exists not only because they both depend on ε_t, but they also both depend on v_{it}. Formally, the application of the Wiener-Kolmogorov formula leads to

$$\mathbb{E}_{it}(x_{it+1}) = \left[L^{-1}g_{x_i,(\theta_i,y)}(L)\left(\Gamma^*(L^{-1})^T\right)^{-1}\right]_+ \Gamma^*(L)^{-1}\begin{pmatrix} \theta_{it} \\ y_t \end{pmatrix}, \tag{A.30}$$

where $g_{x_i,(\theta_i,y)}(L)$ is the variance-covariance generating function between x_i and the information set. Given the equilibrium guess, such function takes the form

$$g_{x_i,(\theta_i,y)}(L) = \left[\, X(L)A(L^{-1})\sigma_\varepsilon^2 + V(L)\sigma_v^2 \quad X(L)(L^{-1}-\lambda)G(L^{-1})\sigma_\varepsilon^2 \,\right]. \tag{A.31}$$

It follows that

$$\begin{aligned} &L^{-1}g_{x_i,(\theta_i,y)}(L)\left(\Gamma^*(L^{-1})^T\right)^{-1} \\ &= \left[L^{-1}\left(V(L)\sigma_v^2 + X(L)\sigma_\varepsilon^2 A(\lambda)\right) \quad \sigma_\varepsilon\sigma_v L^{-1}\frac{1-\lambda L}{L-\lambda}\left(X(L) - V(L)A(\lambda)\right)\right]. \end{aligned} \tag{A.32}$$

The application of the annihilator operator requires to take the annihiland minus the principal part of its Laurent series expansion. All the terms have the usual principal part around $L = 0$. However, the term containing $\frac{1-\lambda L}{L-\lambda}$ also has a principal part around $L = \lambda$, it follows that

$$\begin{aligned} &\left[\left(\frac{1-\lambda L}{L-\lambda}\right)\frac{1}{L}\left(X(L) - V(L)A(\lambda)\right)\right]_+ \\ &\quad = L^{-1}\left[\left(\frac{1-\lambda L}{L-\lambda}\right)\left(X(L) - V(L)A(\lambda)\right) + \frac{1}{\lambda}\left(X(0) - V(0)A(\lambda)\right)\right] \\ &\quad - \frac{1-\lambda^2}{L-\lambda}\frac{1}{\lambda}\left(X(\lambda) - V(\lambda)A(\lambda)\right). \end{aligned} \tag{A.33}$$

Finally one gets

$$\begin{aligned}
&\mathbb{E}_{it}(x_{it+1}) \\
&\quad = L^{-1}\Big[X(L) - X(0)\Big]\varepsilon_t - \big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[X(\lambda) - X(0) - \big(V(\lambda) - V(0)\big)A(\lambda)\Big]\varepsilon_t \\
&\qquad + L^{-1}\Big[V(L) - V(0)\Big]v_{it} + \tfrac{\tau(\lambda)}{A(\lambda)}\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[X(\lambda) - X(0) - \big(V(\lambda) - V(0)\big)A(\lambda)\Big]v_{it}.
\end{aligned} \tag{A.34}$$

STEP 3: FIXED POINT We begin by manipulating condition (10) to get the following relationship between $X(L)$ and $Y(L)$,

$$X(L) = \tilde{\gamma}_y(L)Y(L) + \tilde{\gamma}_\theta(L)A(L), \tag{A.35}$$

where $\tilde{\gamma}_y(L) = -\frac{\gamma_y(L)}{\gamma_x(L)}$, and $\tilde{\gamma}_\theta(L) = -\frac{\gamma_\theta(L)}{\gamma_x(L)}$. Next we substitute the equilibrium guess and expressions (A.28), (A.29), and (A.34) into model (7), which leads to the expression

$$\begin{aligned}
&\phi_x\Bigg[L^{-1}\Big[X(L) - X(0)\Big]\varepsilon_t - \big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[X(\lambda) - X(0) - \big(V(\lambda) - V(0)\big)A(\lambda)\Big]\varepsilon_t \\
&+ L^{-1}\Big[V(L) - V(0)\Big]v_{it} + \tfrac{\tau(\lambda)}{A(\lambda)}\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[X(\lambda) - X(0) - \big(V(\lambda) - V(0)\big)A(\lambda)\Big]v_{it}\Bigg] \\
&+ \phi_y\Big[\big[(L-\lambda)G(L) + \lambda G(0)\big]L^{-1}\varepsilon_t - \big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{(1-\lambda L)}G(0)\varepsilon_t + \tau(\lambda)\tfrac{1-\lambda^2}{(1-\lambda L)}\tfrac{G(0)}{A(\lambda)}v_{it}\Big] \\
&+ \phi_\theta\Big[\big[A(L) - A(0)\big]L^{-1}\varepsilon_t - \big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\big[A(\lambda) - A(0)\big]\varepsilon_t \\
&- \tau(\lambda)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[1 - \tfrac{A(0)}{A(\lambda)}\Big]v_{it}\Big] \\
&= \psi_x(L)\big(X(L)\varepsilon_t + V(L)v_{it}\big) + \psi_y(L)(L-\lambda)G(L)\varepsilon_t + \psi_\theta(L)A(L)\varepsilon_t + \psi_\theta(L)v_{it}.
\end{aligned} \tag{A.36}$$

As one would expect, both on the left and right hand sides there are lag polynomials that multiply ε_t and v_{it}. Because the two stochastic processes are uncorrelated, the equality must hold independently for the terms that multiply ε_t for those that multiply v_{it}. Taking into account relationship (A.35), equation (A.36) thus defines two fixed points: one for $(L-\lambda)G(L)$ and one for $V(L)$. Differently from the full information case, the fixed point for the aggregate y_t (that defined by the terms multiplying ε_t) also contains elements of the function $V(L)$, more precisely the constant $V(0) - V(\lambda)$. Therefore, in order to solve for $(L-\lambda)G(L)$, we need first to solve for $V(L)$. Taking the fixed point condition for the terms that multiply v_{it}, multiplying both sides by L and rearranging one obtains

$$\begin{aligned}
V(L)\phi_x(L) =& \phi_x V(0) - \phi_x \tfrac{\tau(\lambda)}{A(\lambda)}\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\left[X(\lambda) - X(0) - \big(V(\lambda) - V(0)\big)A(\lambda)\right]L \\
&- \tfrac{\tau(\lambda)}{A(\lambda)}\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\left[\phi_y G(0) + \phi_\theta\big(A(\lambda) - A(0)\big)\right]L + \psi_\theta(L)L,
\end{aligned} \tag{A.37}$$

where $\phi_x(L) \equiv \phi_x - \psi_x(L)L$. Similarly, the fixed point for $(L-\lambda)G(L)$ is

$$\begin{aligned}
(L-\lambda)G(L)\Phi(L) =& -\phi_x\tilde{\gamma}_y(0)\lambda G(0) - \phi_x\big(\tilde{\gamma}_\theta(L)A(L) - \tilde{\gamma}_\theta(0)A(0)\big) \\
&+ \phi_x\big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[X(\lambda) - X(0) - \big(V(\lambda) - V(0)\big)A(\lambda)\Big]L \\
&+ \phi_y\Big[\lambda - \big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{(1-\lambda L)}L\Big]G(0) - \phi_\theta\Big[\big[A(L) - A(0)\big] \\
&- \big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\big[A(\lambda) - A(0)\big]L\Big] \\
&+ \psi_x(L)\tilde{\gamma}_\theta(L)A(L)L + \psi_\theta(L)A(L)L,
\end{aligned} \tag{A.38}$$

where we have used (A.35) to substitute for, $X(L) - X(0)$, and, $X(L)$, and, $\Phi(L) \equiv \phi_x(L) + \phi_y - \psi_y(L)L$. The next Lemma will prove very useful.

Lemma A4. $V(\lambda) = \tilde{\gamma}_\theta(\lambda)$.

Proof. Evaluate (A.37) at λ and rearrange to obtain

$$\begin{aligned} V(\lambda)\psi_x(\lambda)\lambda = &- \phi_x \tfrac{\tau(\lambda)}{A(\lambda)} [X(\lambda) - X(0)] - \phi_x\big(1-\tau(\lambda)\big)\big(V(\lambda) - V(0)\big) \\ &- \tfrac{\tau(\lambda)}{A(\lambda)} \big[\phi_y G(0)\lambda + \phi_\theta\big(A(\lambda) - A(0)\big)\big] + \psi_\theta(\lambda)\lambda. \end{aligned} \tag{A.39}$$

Next, evaluate (A.38) at λ and rearrange to obtain

$$\begin{aligned} 0 = &- \tau(\lambda)\phi_x\big(X(\lambda) - X(0)\big) + -\phi_x\big(1-\tau(\lambda)\big)\big(V(\lambda) - V(0)\big)A(\lambda) - \phi_y\tau(\lambda)G(0)\lambda \\ &- \phi_\theta\big(A(\lambda) - A(0)\big)\tau(\lambda) + \psi_x(\lambda)\tilde{\gamma}_\theta(\lambda)A(\lambda)\lambda + \psi_\theta(\lambda)A(\lambda)\lambda. \end{aligned} \tag{A.40}$$

Clearly, for (A.39) and (A.40) to hold, assuming $A(\lambda) \neq 0$, $\psi_x(\lambda) \neq 0$ and $\lambda \neq 0$, it must be that $V(\lambda) = \tilde{\gamma}_\theta(\lambda)$. □

We can now use Lemma A4 to substitute for $V(\lambda)$ in (A.37) and (A.38). It follows that to solve for $(L-\lambda)G(L)$ we just need an expression for $V(0)$, to which we now turn. From Assumption 1, we know that there is a root ζ_V that needs to be removed for $V(L)$ to be stationary. We achieve this by choosing the appropriate constant $V(0)$ so that the numerator on the right hand side of (A.37) vanishes when evaluated at ζ_V,

$$\begin{aligned} &\phi_x V(0) - \phi_x \tfrac{\tau(\lambda)}{A(\lambda)} \tfrac{1-\lambda^2}{\lambda(1-\lambda\zeta_V)} \big[X(\lambda) - X(0) - \big(\tilde{\gamma}_\theta(\lambda) - V(0)\big)A(\lambda)\big]\zeta_V \\ &- \tfrac{\tau(\lambda)}{A(\lambda)} \tfrac{1-\lambda^2}{\lambda(1-\lambda\zeta_V)} \big[\phi_y G(0) + \phi_\theta\big(A(\lambda) - A(0)\big)\big]\zeta_V + \psi_\theta(\zeta_V)\zeta_V = 0 \end{aligned} \tag{A.41}$$

Using (A.35) so substitute for $X(\lambda) - X(0)$, and rearranging one obtain the expression

$$\phi_x V(0)A(\lambda) = m(\lambda)\big(\phi_x\tilde{\gamma}_y(0) + \phi_y\big)\lambda G(0) + n(\lambda), \tag{A.42}$$

where

$$m(\lambda) \equiv \frac{\tau(\lambda)(1-\lambda^2)\zeta_V}{(1-\lambda\zeta_V)\lambda - \tau(\lambda)(1-\lambda^2)\zeta_V}, \tag{A.43}$$

and

$$n(\lambda) \equiv \frac{\phi_\theta\tau(\lambda)(1-\lambda^2)\big(A(\lambda) - A(0)\big)\zeta_V - \phi_x\tau(\lambda)(1-\lambda^2)\tilde{\gamma}_\theta(0)A(0)\zeta_V - \psi_\theta(\zeta_V)\zeta_V\lambda A(\lambda)}{(1-\lambda\zeta_V)\lambda - \tau(\lambda)(1-\lambda^2)\zeta_V}. \tag{A.44}$$

Next we used (A.42) in (A.38), and we also substitute $X(\lambda) - X(0)$ using (A.35) to get

$$(L-\lambda)G(L) = \frac{-\lambda G(0)\big(\phi_x\tilde{\gamma}_y(0) + \phi_y\big)H(L) + J(L)}{\Phi(L)(1-\lambda L)\lambda}, \tag{A.45}$$

where

$$H(L) = \lambda(1-\lambda L) - \big(1-\tau(\lambda)\big)(1-\lambda^2)(1+m(\lambda))L, \tag{A.46}$$

and

$$J(L) = \big(1-\tau(\lambda)\big)(1-\lambda^2)\big[n(\lambda) - \phi_x\tilde{\gamma}_\theta(0)A(0) + \phi_\theta\big(A(\lambda)-A(0)\big)\big]L \\ + A(0)\big(\phi_x\tilde{\gamma}_y(0)+\phi_y\big)\lambda(1-\lambda L) \\ - \big[\big(\phi_x - \psi_x(L)L\big)\tilde{\gamma}_\theta(L) + \phi_\theta - \psi_\theta(L)L\big]A(L)\lambda(1-\lambda L). \tag{A.47}$$

Under Assumption 1, $\Phi(L)$ has a zero inside the unit circle at ζ, which means that we need to choose the constant $G(0)$ so to cancel it. This is achieved by setting

$$-\lambda G(0)\big(\phi_x\tilde{\gamma}_y(0)+\phi_y\big)H(\zeta) + J(\zeta) = 0. \tag{A.48}$$

Solving for $G(0)$ and substituting back into (A.45) one gets

$$(L-\lambda)G(L) = \frac{J(L)H(\zeta) - J(\zeta)H(L)}{\Phi(L)(1-\lambda L)\lambda}. \tag{A.49}$$

Next, recall that we defined

$$\xi_y(L) \equiv A(0)\big(\phi_x\tilde{\gamma}_y(0)+\phi_y\big) - \big[\big(\phi_x-\psi_x(L)L\big)\tilde{\gamma}_\theta(L) + \phi_\theta - \psi_\theta(L)L\big]A(L), \tag{A.50}$$

and letting

$$\tilde{\xi} \equiv n(\lambda) - \phi_x\tilde{\gamma}_\theta(0)A(0) + \phi_\theta\big(A(\lambda)-A(0)\big), \tag{A.51}$$

one can show that (A.49) can be written as

$$(L-\lambda)G(L) = \frac{\xi_y(\zeta)-\xi_y(L)}{\Phi(L)} - \big(1-\tau(\lambda)\big)(1-\lambda^2)(\zeta-L)\frac{\tilde{\xi}-(1+m(\lambda)\xi_y(\zeta)}{H(\zeta)\Phi(L)(1-\lambda L)}. \tag{A.52}$$

Using the factorization $\Phi(L) = (\zeta - L)\tilde{\Phi}(L)$, and defining

$$\mathcal{A}(\lambda) \equiv \frac{\tilde{\xi} - \big(1+m(\lambda)\big)\xi_y(\zeta)}{H(\zeta)}, \tag{A.53}$$

expression (19) follows. Finally, for the solution to be consistent with the information set that we have used to derive it, it must be that the polynomial in (19) vanishes at $L=\lambda$, which corresponds to condition (20) in the Theorem.

The last step of the proof consists in making sure that when the equilibrium coefficients are evaluated using the λ that solves (20), there are no other points at which $Y(L)$ vanishes inside the unit circle. More precisely, it has to be that there is no $\xi \neq \lambda$ that solves

$$\mathcal{Y}(\xi)\tilde{\Phi}(\xi) = (1-\tau(\lambda))(1-\lambda^2)\frac{\mathcal{A}(\lambda)}{1-\lambda\xi}, \tag{A.54}$$

such that $|\xi| \in (-1,1)$. If this was not the case, then the information conveyed by y_t in equilibrium would be inconsistent with the information used to derive the expectations that we use to determine the fixed point. More precisely, the factorization of $\Gamma(L)$ would be incorrect, as $\Gamma^*(L)$ in (A.24) would still be non-invertible. To see this, suppose that λ is a solution to (20), while ξ is a solution to (A.54), and they are both inside the unit circle. Then, the equilibrium function must have the form $\tilde{G}(L)(L-\lambda)(L-\xi)$, but the factorization above only removes the zero associated with λ. It follows that

$$\Gamma^*(L) = \frac{1}{\sqrt{A(\lambda)^2\sigma_\varepsilon^2+\sigma_v^2}}\begin{bmatrix} A(L)A(\lambda)\sigma_\varepsilon^2+\sigma_v^2 & \sigma_\varepsilon\sigma_v\frac{1-\lambda L}{L-\lambda}\big(A(L)-A(\lambda)\big) \\ A(\lambda)\sigma_\varepsilon^2(L-\lambda)(L-\xi)\tilde{G}(L) & \sigma_\varepsilon\sigma_v\tilde{G}(L)(L-\xi)(1-\lambda L)\end{bmatrix}, \tag{A.55}$$

whose determinant still vanishes at $L = \xi$, so that $\Gamma^*(L)$ is not the appropriate factorization. In this case one can modify the initial guess and consider $N > 1$ roots inside the unit circle, looking then for a condition like (20) to deliver exactly N solutions. We restrict our attention to $N = 1$ for simplicity and because the full description of the space of REE with confounding dynamics is beyond the scope of this paper, but we hope it is clear that our methods extend to the more general case.

Step 4: No Information from the Model The last thing to check to complete the proof is to ensure that there is no information that is transmitted by a clever manipulation of the model conditions – which are part of the information set of the agents – combined with the knowledge of the history of θ_{it} and y_t. For instance, suppose that the market clearing condition (10) is specified so that $\int_0^1 x_{it}\mu(i)di = y_t$, which means that y_t is the aggregate of x_{it}, then this would imply $X(L) = Y(L)$, which would result in $x_{it} - y_t = V(L)v_{it}$. Because rational agents know all this, they know that the difference $x_{it} - y_t$ is just a linear combination of the individual innovations v_{it}. It follows that they could, in principle, back out the realizations of v_{it}'s by inverting $V(L)$. More generally, the link between $X(L)$ and $Y(L)$ due to (10) can be used by rational agents to obtain additional information on the underlying innovations. For this not to happen, if one augments the information set of the agents by $x_{it} - y_t$, the information matrix must still be non-invertible at λ. The following Lemma shows that this is indeed the case for the equilibrium of Theorem 1.

Lemma A5. *In the equilibrium with confounding dynamics of Theorem 1, consider the augmented information matrix $\tilde{\Gamma}(L)$, where*

$$\begin{pmatrix} \theta_{it} \\ y_t \\ x_{it} - y_t \end{pmatrix} = \tilde{\Gamma}(L)\begin{pmatrix} \varepsilon_t \\ v_{it} \end{pmatrix} = \begin{bmatrix} A(L) & 1 \\ Y(L) & 0 \\ X(L) - Y(L) & V(L) \end{bmatrix}\begin{pmatrix} \varepsilon_t \\ v_{it} \end{pmatrix}. \tag{A.56}$$

The 2-by-2 minors of $\tilde{\Gamma}(L)$ all vanish at λ.

Proof. Matrix $\tilde{\Gamma}(L)$ has three minors, whose determinants are, respectively, $Y(L)$, $Y(L)V(L)$, and, $A(L)V(L) - (X(L) - Y(L))$. The first two minors clearly vanish at λ since, by construction, $Y(\lambda) = 0$. For the third minor, use (A.35) to write

$$A(L)V(L) - (X(L) - Y(L)) = A(L)V(L) - \tilde{\gamma}_y(L)Y(L) - \tilde{\gamma}_\theta(L)A(L) + Y(L). \tag{A.57}$$

We thus need to show that

$$A(\lambda)V(\lambda) = \tilde{\gamma}_\theta(\lambda)A(\lambda), \tag{A.58}$$

but this follows immediately from Lemma A4. □

A.3. Derivation of full information solution of RBC model

In terms of the notation we used in Appendix A.1, $\mathcal{Y}(L)$ would correspond to $\mathcal{R}(L)$ in the application of Section 5, and $\mathcal{X}(L)$ to $\mathcal{K}(L)$. We first note that here $\gamma_x(L) = (1 - \alpha)L$, which means $\gamma_x(0) = 0$, we are thus forced to take the alternative route described in Appendix A.1 and solve first for $\mathcal{K}(L)$ and leave the solution to $\mathcal{R}(L)$ as a straightforward corollary. Under full information we know that $\mathbb{E}_{it}(k_{it+2}) = \left[\mathcal{K}(L) - \mathcal{K}(0)\right]L^{-1}\varepsilon_t + \left[\mathcal{V}(L) - \mathcal{V}(0)\right]L^{-1}v_{it}$, $\mathbb{E}_{it}(k_{t+1}) = k_{t+1} = \mathcal{K}(L)\varepsilon_t$, and $\mathbb{E}_{it}(a_{it+1}) = \mathbb{E}_{it}(a_{t+1}) = \left[A(L) - A(0)\right]L^{-1}\varepsilon_t$. Substituting

(30) into (31), using the above expressions for the expectations, aggregating over agents, multiplying both sides by L, and rearranging, one obtains the fixed point condition

$$\mathcal{K}(L) = \frac{\alpha\beta\mathcal{K}(0) + \big(1 - \eta(1-\alpha\beta) - L\big)A(L) - A(0)}{\alpha(\zeta - L)(\beta/\zeta - L)}. \tag{A.59}$$

To ensure stationarity we choose $\mathcal{K}(0) = \frac{1}{\alpha\beta}\big(A(0) - \big(1 - \eta(1-\alpha\beta) - \zeta\big)A(\zeta)\big)$. Next substitute this expression in (A.59), and specify $A(L) = \frac{1}{1-\rho L}$. By construction the denominator polynomial contains the factor $(\zeta - L)$, which can be easily isolated and simplified with the same factor at the denominator, so to finally obtain

$$\mathcal{K}(L) = \frac{\frac{\zeta}{\alpha\beta}\left(\frac{1-(1-\eta(1-\alpha\beta))\rho}{1-\rho\zeta}\right)}{(1-\rho L)(1-\frac{\zeta}{\beta}L)}. \tag{A.60}$$

Evaluating the characteristic polynomial (34) at 1 one can show that, $\alpha(\zeta-1)(\beta/\zeta-1)/(1-\alpha) = \eta(1-\alpha\beta)$. Adding, $\zeta - 1$, on both sides and rearranging one can show that, $\alpha(1-\zeta)(\alpha\beta/\zeta-1)/(1-\alpha) = \eta(1-\alpha\beta) - 1 + \zeta$. Now take the term $\frac{1-(1-\eta(1-\alpha\beta))\rho}{1-\rho\zeta}$, add and subtract $\rho\zeta$ at the numerator, to obtain

$$\mathcal{K}(L) = \frac{\frac{\zeta}{\alpha\beta}(1+\kappa)}{(1-\rho L)(1-\frac{\zeta}{\beta}L)}, \tag{A.61}$$

where $\kappa \equiv \frac{\rho(1-\zeta)(\alpha\beta/\zeta-1)}{(1-\rho\zeta)(1-\alpha)}$. It can be showed that $\kappa = 0$ for $\eta = 1$, which corresponds to the case of logarithmic preferences, and $\kappa > 0$ (resp. < 0) when $\eta < 1$ (resp. > 1). The expression for $\mathcal{R}(L)$ can be obtained using the relationship, $\mathcal{R}(L) = A(L) - (1-\alpha)\mathcal{K}(L)L$.

A.4. *Proof of Corollary 1*

The proof of the corollary is a straightforward application of the following lemma.

Lemma A6. *Consider the Real Business Cycle model (30)-(31). Let the information sets be specified as in (37). There exists a Rational Expectations Equilibrium with Confounding Dynamics of the form, $k_{t+1} = K(L)\varepsilon_t$, and $r_t = R(L)\varepsilon_t$, with*

$$K(L) = \mathcal{K}(L) - \big(1-\tau(\lambda)\big)(1-\lambda^2)\frac{\mathcal{A}_k(\lambda)}{(1-\lambda L)(\tilde{\zeta} - L)}, \tag{A.62}$$

and, $R(L) = A(L) - (1-\alpha)K(L)L$, if there exists a $\lambda \in (-1, 1)$, that solves

$$\mathcal{R}(\lambda)(\lambda - \tilde{\zeta}) = (1-\alpha)\big(1-\tau(\lambda)\big)\mathcal{A}_k(\lambda)\lambda, \tag{A.63}$$

where $\mathcal{K}(L)$ and $\mathcal{R}(L)$ are the full information solutions, $\tau(\lambda) \equiv \frac{A(\lambda)^2\sigma_\varepsilon^2}{A(\lambda)^2\sigma_\varepsilon^2+\sigma_v^2}$, $\mathcal{A}_k(\lambda)$ is a function of λ that depends only on exogenous parameters, and $R(L)$ has a zero inside the unit circle equal to λ.

Proof. The proof follows the same steps as that of Theorem 1, with the difference that we solve for $X(L)$ first – $K(L)$ in the application. Recall that

$$\phi_x = \alpha\beta, \quad \phi_y = 1-\alpha\beta, \quad \phi_\theta = 1, \quad \psi_x(L) = \alpha(1+\beta) - \alpha L, \quad \psi_y(L) = 0, \quad \psi_\theta(L) = -1,$$

and

$$\gamma_x(L) = (1-\alpha)L, \quad \gamma_y(L) = 1, \quad \gamma_\theta(L) = -1.$$

Note that, although the notation adopted in the model has the two variables having different time subscripts, r_t and k_{t+1}, they are both pre-determined at time t, and so they are both functions of possibly the infinite history of ε_t up to time t. Since we are looking for an equilibrium with confounding dynamics, we operationalize the condition $R(\lambda) = 0$ by conjecturing

$$R(L) = (L-\lambda)G(L), \tag{A.64}$$

where $G(L)$ has no zeros inside the unit circle. Because in equilibrium $R(L) = A(L) - (1-\alpha)K(L)L$, the conjecture immediately implies

$$A(\lambda) = (1-\alpha)K(\lambda)\lambda, \tag{A.65}$$

a relationship that will be useful in what follows. One important remark on (A.65) is that it implies $\lambda \neq 0$. In fact, evaluating the expression at $\lambda = 0$, provided that $K(0)$ is well defined, which must be the case in the solution we want to characterize, gives $A(0) = 0$, which never holds by assumption. Hence, the statement of the Proposition requires $|\lambda| \in (0, 1)$. The information set takes the form of (A.22), where $x_{it} = a_{it}$ and $y_t = r_t$, so that $\mathbb{E}_{it}(a_{t+1})$ and $\mathbb{E}_{it}(r_{t+1})$ are provided by (A.28) and (A.29), respectively. For the term $\mathbb{E}_{it}\big(k_{it+2}\big)$ things require some extra steps. We work under the conjecture that

$$k_{it+1} = K(L)\varepsilon_t + V(L)v_{it}. \tag{A.66}$$

Next, we evaluate the variance-covariance generating function between the information set and k_{it+1}, which is

$$g_{k_i,(a_i,r)}(z) = \left[\, K(z)A(z^{-1})\sigma_\varepsilon^2 + V(z)\sigma_v^2 \quad K(z)(z^{-1}-\lambda)G(z^{-1})\sigma_\varepsilon^2 \,\right]. \tag{A.67}$$

We then use this expression, together with the canonical factorization $\Gamma^*(z)$ in (A.24) in the Wiener-Kolmogorov formula (C.31), and following steps similar to (A.32) and (A.33) to finally get

$$\begin{aligned}
\mathbb{E}_{it}(k_{it+2}) = {} & L^{-1}\Big[K(L) - K(0)\Big]\varepsilon_t - \big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[K(0) - K(\lambda) \\
& - \big(V(0) - V(\lambda)\big)A(\lambda)\Big]\varepsilon_t \\
& + L^{-1}\Big[V(L) - V(0)\Big]v_{it} - \tau(\lambda)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[\tfrac{K(0)-K(\lambda)}{A(\lambda)} + \big(V(0) - V(\lambda)\big)\Big]v_{it}.
\end{aligned} \tag{A.68}$$

We can now use the expressions for the expectational terms to obtain a fixed point condition similar to (A.36),

$$\begin{aligned}
& \alpha(1+\beta)K(L)\varepsilon_t + \alpha(1+\beta)V(L)v_{it} \\
& = \alpha\beta L^{-1}\Big[K(L) - K(0)\Big]\varepsilon_t + \alpha\beta L^{-1}\Big[V(L) - V(0)\Big]v_{it} \\
& - \alpha\beta\big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[K(\lambda) - K(0) - \big(V(\lambda) - V(0)\big)A(\lambda)\Big]\varepsilon_t \\
& + \alpha\beta\tfrac{\tau(\lambda)}{A(\lambda)}\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[K(\lambda) - K(0) - \big(V(\lambda) - V(0)\big)A(\lambda)\Big]v_{it} \\
& + \alpha K(L)L\varepsilon_t + \alpha V(L)Lv_{it} + A(L)\varepsilon_t + v_{it} - \big[A(L) - A(0)\big]L^{-1}\varepsilon_t
\end{aligned}$$

$$
\begin{aligned}
&+\big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\big[A(\lambda)-A(0)\big]\varepsilon_t \\
&-\tfrac{\tau(\lambda)}{A(\lambda)}\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}\Big[A(\lambda)-A(0)\Big]v_{it}+(1-\alpha\beta)\Big[A(L)-(1-\alpha)K(L)L-A(0)\Big]L^{-1}\varepsilon_t \\
&+(1-\alpha\beta)\big(1-\tau(\lambda)\big)\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}A(0)\varepsilon_t-(1-\alpha\beta)\tfrac{\tau(\lambda)}{A(\lambda)}\tfrac{1-\lambda^2}{\lambda(1-\lambda L)}A(0)v_{it},
\end{aligned}
\tag{A.69}
$$

where we have used $(L-\lambda)G(L)=A(L)-(1-\alpha)K(L)L$, and thus $-\lambda G(0)=A(0)$, to substitute for terms related to $G(L)$. The fixed point equation contains only terms related to the endogenous polynomials $V(L)$ and $K(L)$, and one can proceed to solve for the fixed point as in the proof of Theorem 1. In particular, using the same steps as in Lemma A4, one can show that $A(\lambda)V(\lambda)=K(\lambda)$, and, in addition, we know that (A.65) holds, so we can set $K(\lambda)=\frac{A(\lambda)}{\lambda(1-\alpha)}$. The uniqueness of a stationary solution under Assumption 1 and condition (34), is once again obtained by the appropriate choice of $V(0)$ and $K(0)$. In the end, the expression for $\mathcal{A}_k(\lambda)$, analogue to the constant $\mathcal{A}(\lambda)$ in Theorem 1, can be simplified to

$$
\mathcal{A}_k(\lambda)=\frac{\big[(1-\lambda\beta)\lambda-\tau(\lambda)(1-\lambda^2)\beta\big](1-\beta)A(\lambda)+\beta\big(\eta(1-\beta\alpha)-1\big)\big(1-\tau(\lambda)\big)(1-\lambda^2)A(0)+(1-\lambda\beta)\lambda\big(\eta(1-\beta\alpha)-1+\zeta\big)A(\zeta)}{\lambda\alpha(1+\beta)\big[(1-\lambda\beta)(\zeta-\lambda)-\tau(\lambda)(1-\lambda^2)(\zeta-\beta)\big]}
\tag{A.70}
$$

The condition for the existence of one $|\lambda|\in(0,1)$ follows from using $K(L)$ to write $R(L)$ and then imposing $R(\lambda)=0$. The same argument that we have used in the proof of Theorem 1 to argue that when the equilibrium coefficients are evaluated using the λ that solves $R(\lambda)=0$, there must be no other points at which $R(L)$ vanishes inside the unit circle, applies here as well. This completes the proof. □

The proof of Corollary 1 consists in plugging $A(L)=\frac{1}{1-\rho L}$ into the above expressions and rearranging terms when possible.

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jet.2021.105251.

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